Logarithmic Torsion Homology Growth and Computability

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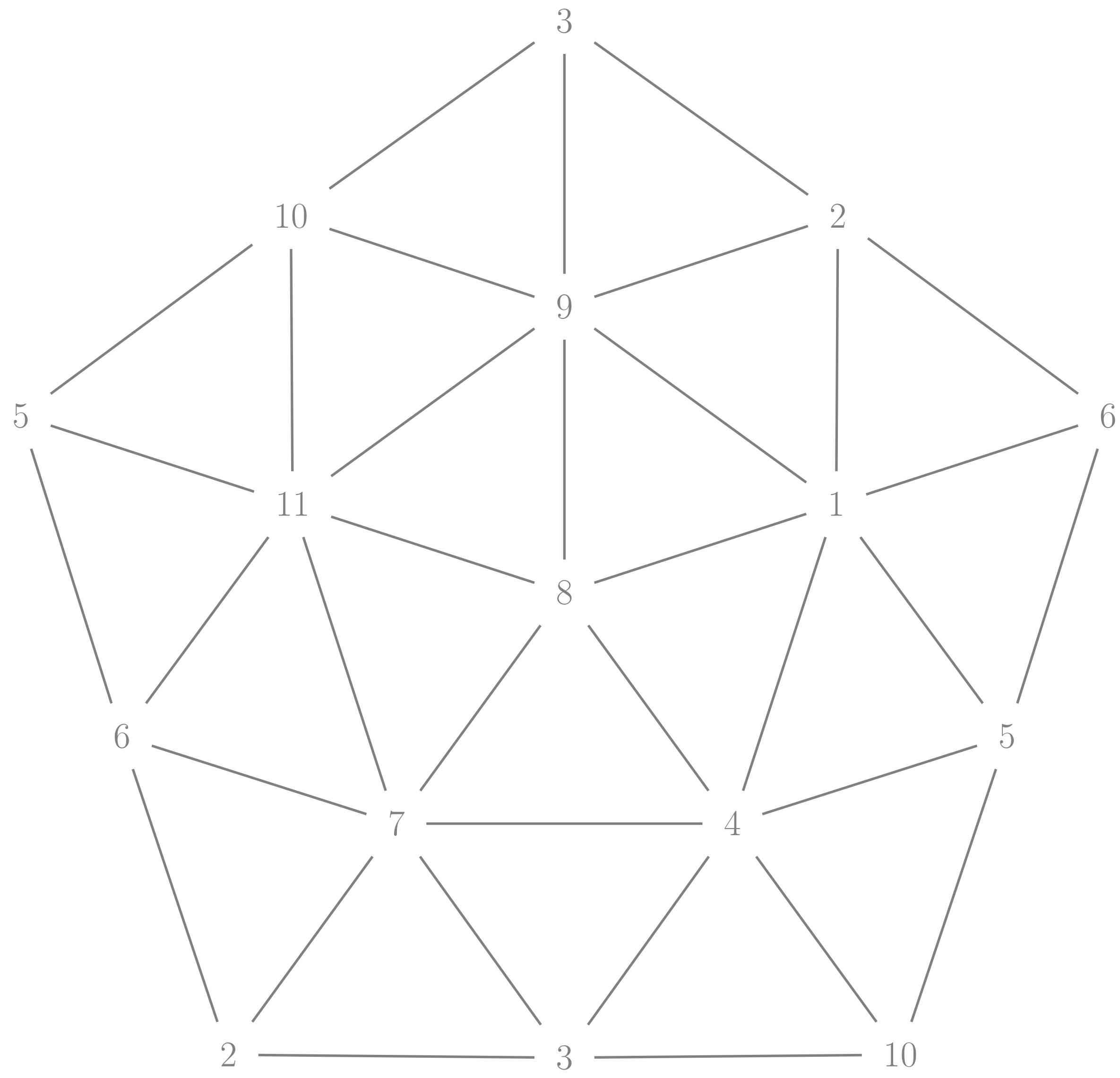

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Life is growth. If we stop growing, technically and spiritually, we are as good as dead.

– Morihei Ueshiba

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0. Introduction

Algebraic topology is concerned with attaching homotopy-invariant algebraic gadgets to topological spaces. Let us apply a well-studied construction, (integral) singular homology $H_*(-;\mathbb{Z})$, to a particularly tame topological space, a finite CW-complex X. This provides us with a family of finitely-generated $\mathbb{Z}$-modules

$$H_*(X;\mathbb{Z}) \cong_{\mathbb{Z}} \text{free part} \oplus \text{torsion part},$$

that encode information about X by mapping simplicies into it. For fixed $n \in \mathbb{N}$, this gives a sensible way to "quantify" the n-dimensional topological complexity of X, by reading off the rank of the free part of $H_n(X;\mathbb{Z})$ (called the *n-th Betti number $b_n(X;\mathbb{Q})$ of X*) and the logarithm of the cardinality of the torsion subgroup of $H_n(X;\mathbb{Z})$ (denoted $\log \operatorname{tors} H_n(X;\mathbb{Z})$).

Assembling all the Betti numbers of X into an alternating sum gives rise to the *Euler characteristic* $\chi(X)$, which has numerous applications and satisfies the desirable property of being *multiplicative*: given a finite d-sheeted covering map $f : Y \to X$, Y inherits the structure of a finite CW-complex and the value of χ simply scales in the index: $\chi(Y) = d \cdot \chi(X)$. This property simplifies computations and is one of the strenghts of working with the Euler characteristic. Unfortunately, it fails dramatically for Betti numbers and the cardinality of the torsion subgroups: there exist self-coverings of circles of arbitrary degree. We wish to rectify this by artificially producing an invariant that is multiplicative, and our strategy of choice will be to pass to an asymptotic setting. Before we start, let us phrase the constructions above in the language of groups, which can be done by essentially passing to the fundamental groups.

Let G be a group which admits a *finite classifying space* X, that is, a finite path-connected CW-complex with fundamental group isomorphic to G and contractible universal covering. Then all of our considerations above translate into a group-theoretic setting: singular homology of X translates to *group homology $H_*(G;\mathbb{Z})$ of G*, and finite coverings of X correspond to finite index subgroups of G. We express our refinement of $b_n(-;\mathbb{Q})$ and $\log \operatorname{tors} H_n(-;\mathbb{Z})$ to a multiplicative invariant more generally. Let I be an arbitrary numerical group invariant. For a group G, we define a *residual chain in G* to be a nested sequence $(N_k)_k$ of finite index normal subgroups of G with trivial intersection. We define the associated *invariant growth* *(with respect to the chain $(N_k)_k$)* as

$$\widehat{I}\big(G,(N_k)_k\big) := \limsup_{k\to\infty} \frac{I(N_k)}{[G:N_k]}.$$

A priori, this definition depends on the chain $(N_k)_k$. Now assume that $H \supseteq N_{k_0}$ is a finite index subgroup of G that eventually contains the chain $(N_k)_k$. Then one checks easily that

$$\widehat{I}\big(G,(N_k)_k\big) = [G:H]\cdot \widehat{I}\big(H,(N_k)_{k\geq k_0}\big),$$

where $(N_k)_{k\geq k_0}$ is a residual chain of H defined by capping off $(N_k)_k$ at the k_0-th term. Forcing multiplicativity in this manner raises questions:

- Which groups admit such residual chains?
- Does $\widehat{I}\big(G,(N_k)_k\big)$ depend on the choice of residual chain and, if yes, how?
- What are interesting invariants I to apply this to?
- What interesting properties of I are passed on to $\widehat{I}$?

Discussing these questions for homological invariants is the main focus of this thesis. Applying our growth construction above to $b_n(G;\mathbb{F})$ (a version of the Betti number with coefficients) and $\log \operatorname{tors} H_n(G;\mathbb{Z})$ leads to the notions of *$\mathbb{F}$-Betti number growth* $\widehat{b}_n\,(G,N_*;\mathbb{F})$ and *torsion growth* $\widehat{t}_n\,(G,N_*)$. Intriguingly, these two facets of homology growth reflect two extremes in the study of growth of invariants.

By *Lück's approximation theorem*, $\mathbb{Q}$-Betti number growth is a chain-independent proper limit and coincides with the *L^2-Betti number*, which is an analytical invariant that could also be viewed as a way to cook up an equivariant version of the Betti numbers that is multiplicative under finite index subgroups.

On the other hand, there is very limited general knowledge about torsion growth, with no known examples where it depends on the chain or is not a proper limit. Torsion growth vanishes for many groups, and the first group where it was computed to be positive is relatively recent.

The *$\mathbb{F}_p$-Betti number growth* and *rank gradient* lie somewhere between these poles, making the study of this intermediary case interesting for understanding more about the overall spectrum of general growths of invariants.

Conventions

The natural numbers $\mathbb{N}$ contain 0. We have $\log(\infty)=\infty$, $\infty+\infty=\infty$, $\infty^{-1}=0$ and so on. All modules are left-modules, and we use the terms "Abelian group" and "$\mathbb{Z}$-module" interchangeably. $H_n(X) := H_n(X;\mathbb{Z})$. If G is a group, we write $|G| \in \mathbb{N}\cup\{\infty\}$. Prime numbers are chosen to be positive.

Overview

The following theorems are reformulated and simplified with minor adjustments to streamline their presentation.

In Chapter 1, we familiarize ourselves with *residually finite groups*, which are the groups that admit residual chains and thus form the basis for the study of invariant growth. We then proceed to define general growth of invariants of groups, and observe the generic properties that they share. We prove the multiplicativity criterion alluded to in the introduction, and that additivity of group invariants under free products is inherited to the associated invariant growth:

Theorem 1.52. *Let I be a numerical invariant of groups such that for all residually finite groups G_1 and G_2 one has $I(G_1 * G_2) = I(G_1) + I(G_2)$. Let G_1 and G_2 be residually finite groups such that $\widehat{I}(G_1)$ and $\widehat{I}(G_2)$ are chain-independent. Then the same holds for $G_1 * G_2$ and we have*

$$\widehat{I}(G_1 * G_2) = \widehat{I}(G_1) + \widehat{I}(G_2) + I(\mathbb{Z}) \cdot \big(1 - |G_1|^{-1} - |G_2|^{-1} + |G_1 * G_2|^{-1}\big).$$

In particular, we have that:

- $\widehat{t_n}\,(G_1 * G_2) = \widehat{t_n}\,(G_1) + \widehat{t_n}\,(G_2)$ *for all* $n \in \mathbb{N}$,
- $\widehat{b_1}\,(G_1 * G_2; \mathbb{F}) = \widehat{b_1}\,(G_1; \mathbb{F}) + \widehat{b_1}\,(G_2; \mathbb{F}) + 1 - |G_1|^{-1} - |G_2|^{-1} + |G_1 * G_2|^{-1}$,
- $\widehat{b_n}\,(G_1 * G_2; \mathbb{F}) = \widehat{b_n}\,(G_1; \mathbb{F}) + \widehat{b_n}\,(G_2; \mathbb{F})$ *for all* $n \in \mathbb{N}_{\geq 2}$.

We also discuss the growth of the rank and deficiency of groups, giving rise to the *rank gradient* and *deficiency gradient*, respectively. We showcase how the submultiplicativity of the rank allows to remove the ambiguity of choosing a chain by passing to an *absolute* version of growth.

In Chapter 2, we define group homology and discuss both the algebraic and the topological viewpoint. We introduce the notion of classifying spaces and review finiteness properties of groups. We discuss the universal coefficient theorem and use it to relate Betti numbers with different field coefficients, and Betti numbers with torsion invariants. We then prove the important fact that in the presence of a suitable finiteness property, the numerical invariants associated to group homology of finite index subgroups stay "controlled" in the index. This justifies the logarithm in the definition of torsion growth and seems to be a well-known but yet unrecorded fact to authors in the field.

Theorem 2.60. *Let $N \in \mathbb{N}$ and let G be a group of type FP_{N+1}. For every $k \in \{0, \ldots, N\}$ there exists a constant $d_k(G) \in \mathbb{R}_{\geq 0}$ such that for every finite index subgroup $H \subseteq G$:*

$$\log \operatorname{tors} H_k(H; \mathbb{Z}) \leq d_k(G) \cdot [G : H].$$

In Chapter 3, we finally introduce torsion growth and Betti number growth and put our observations from the previous chapters to use. Concretely, we establish that a discrepancy between $b_n(G;\mathbb{Q})$ and $b_n(G;\mathbb{F}_p)$ for a prime p witnesses positive torsion growth in degrees n or $n-1$ of G, a fact that was used by Avramidi–Okun–Schreve to exhibit the first known example of a finitely presented group with non-zero torsion growth. We compute examples of Betti number growth and torsion growth, and we state Lück's approximation theorem and some of its consequences. We then introduce the class of *right-angled Artin groups* and survey the results which relate their homology growth to the homology of their underlying flag complex. As flag complexes can essentially produce any reasonable sequence of Abelian groups as their homology groups, this leads to realizability results. By observing how direct products of right-angled Artin groups affect the underlying flag complexes, we get an intuition for what kind of behavior we might expect from homology growth of direct products of general groups. With this in mind, we proceed to examine group homology of direct products via the *Künneth theorem*. For Betti number growth, this leads to a Künneth formula:

Theorem 3.82. *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G_1 and G_2 be residually finite groups of type* FP_n. *Let $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ be residual chains, such that $\widehat{b}_i\left(G_l, N_*^l; \mathbb{F}\right)$ are proper limits for $l \in \{1,2\}$ and $i \in \{1,\dots,n\}$. Then the same holds for $\widehat{b}_n\left(G_1 \times G_2, N_*^1 \times N_*^2; \mathbb{F}\right)$ and we have*

$$\widehat{b}_n\left(G_1 \times G_2, N_*^1 \times N_*^2; \mathbb{F}\right) = \sum_{i+j=n} \widehat{b}_i\left(G_1, N_*^1; \mathbb{F}\right) \cdot \widehat{b}_j\left(G_2, N_*^2; \mathbb{F}\right).$$

The situation for torsion growth is more involved, since torsion in tensor products and Tor-terms depends delicately on the concrete structure of the torsion modules involved. We establish upper and lower bounds, which allow for some structural observations.

Theorem 3.88. *Let $n \in \mathbb{N}$, and let G_1 and G_2 be residually finite groups of type* FP_{n+1}, *and let $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$. Assume that $\widehat{t}_i\left(G_1, N_*^1\right)$ and $\widehat{t}_i\left(G_2, N_*^2\right)$ are proper limits for all $i \in \{1,\dots,n\}$. Then*

$$\widehat{t}_n\left(G_1 \times G_2, N_*^1 \times N_*^2\right) \geq \sum_{i+j=n} \Big(b_j^{(2)}(G_2) \cdot \widehat{t}_i\left(G_1, N_*^1\right) + b_i^{(2)}(G_1) \cdot \widehat{t}_j\left(G_2, N_*^2\right)\Big).$$

The upper bound given in Theorem 3.90 is less pretty than the lower bound above and contains many additional error terms. Denote by $\mathbf{T}_n$ the class of residually finite groups with chain-independent vanishing torsion growth up to degree n. Using the upper bound, we obtain the following inheritance of vanishing torsion growth under direct products.

Corollary 3.94. *Let $n \in \mathbb{N}$, let G_1 and G_2 be residually finite groups of type* FP_{n+1}. *If $G_1, G_2 \in \mathbf{T}_{n-1}$ and $\widehat{t}_m\,(G_1 \times G_2)$ is chain-independent for $m \leq n$, then $G_1 \times G_2 \in \mathbf{T}_n$.*

In Chapter 4, we will touch on algorithmic aspects of growth of invariants, specifically decision problems and the problem of realizing real numbers as the invariant growth of a group. We show that the vanishing problem of the (absolute) rank gradient is undecidable, showcasing the importance of understanding invariants under free products and direct products, and of dealing with the dependency on the chain.

Theorem 4.7. *The following algorithmic problem is undecidable:*

> *Given a finite presentation $\langle S \mid R \rangle$, decide whether $\langle S \mid R \rangle$ has vanishing (absolute) rank gradient or not.*

Finally, we combine the surveyed theorems about homology growth of right-angled Artin groups, the fact that flag complexes can realize any sequence of prescribed Abelian groups as its homology groups, and the result about invariant growth of finite-index subgroups, to prove that certain families of real numbers arise as the homological invariant growth of a group with respect to a particular chain.

Corollary 4.11. *Let $(b_i)_{i \geq 1}$ and $(t_i)_{i \geq 2}$ be sequences with values in $\mathbb{Q}_{\geq 0}$ and $\frac{\log \mathbb{N}_{\geq 1}}{\mathbb{N}_{\geq 1}}$, respectively, such that only finitely many values are non-zero. Let $b_0 = t_0 = t_1 = 0$. Then there exists a finitely presented residually finite group G together with a residual chain $N_* \in \mathcal{R}(G)$ such that for all $n \in \mathbb{N}$*

$$\widehat{b}_n\,(G, N_*; \mathbb{Q}) = b_n \quad \text{and} \quad \widehat{t}_n\,(G, N_*) = t_n.$$

While the results of this thesis do not resolve the fundamental issue of (homological) invariant growth depending on residual chains, nor the general lack of understanding regarding positive torsion growth, they do offer an advantage: the proof methods remain entirely within the framework of the growth of invariants. This may prove useful in the future – either as more classes of groups with chain-independent invariant growths are discovered, or as interest shifts to the growth of other types of invariants.

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1. Growth of invariants

In this first chapter we set the stage for the study of growth of invariants of groups. The goal is to stabilize numerical invariants along suitable sequences of finite index subgroups. In Chapter 1.1, we introduce *residually finite groups*, which are groups that admit particularly rich sequences of subgroups. We show that many basic group-theoretic constructions preserve residual finitenes, and that the index of these sequences tends to the cardinality of the group. In Chapter 1.2 we proceed to define *invariant growth*, and we observe some general properties. We discuss to what extent this resembles a subgroup-multiplicative version of the original invariant. Furthermore, we show that additivity under free products of groups is passed on to the growth of the invariant, which will apply in particular to the growth of homological invariants later on. Finally, we introduce the growth of the *rank* of a group. We show that it can be expressed as the *absolute growth* of the rank, which allows us to remove the hassle of dealing with choices of residual chains.

Overview of this chapter.

1.1. Residually finite groups

The main protagonist is a class of groups, which turns out to provide a natural setting for growth of group invariants. The key requirement is that finite index normal subgroups contain enough information about the full group. For a group G, let us denote by $\mathcal{F}(G)$ the set of finite index subgroups. A good reference for an introduction to residually finite groups is the book by Ceccherini-Silberstein and Coornaert [CC10]. We start by recalling two basic theorems in group theory.

Lemma 1.1 (normal core [Rot12, Exercise 2.37.(i)])**.** *Let G be a group, and let $H \subseteq G$ be a subgroup. Let $N := \bigcap_{g \in G} gHg^{-1}$ be the* normal core *of H in G. Then, N is a normal subgroup of G, and N is contained in H. Moreover, if $[G : H] < \infty$ then also $[G : N] < \infty$.*

Theorem 1.2 (isomorphism theorems [Rot09, Chapter 2.6])**.** *Let G be a group. Let $H \subseteq G$ be a subgroup and let $N \trianglelefteq G$ be a normal subgroup. Then*

1. *$SN \subseteq G$ is a subgroup,*
2. *$N \trianglelefteq SN$ is a normal subgroup,*
3. *$S \cap N \trianglelefteq S$ is a normal subgroup,*
4. *the groups $(SN)/N$ and $S/(S \cap N)$ are isomorphic.*

Theorem 1.3 (residually finite group, residual chain)**.** *Let G be a countable group. We call G* residually finite *if one of the following equivalent conditions is satisfied:*

1. *For every $g \in G \backslash \{1\}$ there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(g) \neq 1$;*
2. *For every $g, h \in G$ with $g \neq h$ there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(g) \neq \phi(h)$;*
3. *For every $g \in G \setminus \{1\}$ there exists a finite index normal subgroup $N \trianglelefteq G$ such that $g \notin N$;*
4. *the intersection of all finite index subgroups of G is trivial;*
5. *the intersection of all finite index normal subgroups of G is trivial;*
6. *there exists a* residual chain *in G, i.e., a sequence $(N_k)_{k \in \mathbb{N}}$ of finite index normal subgroups of G, such that for every $k \in \mathbb{N}$ we have $N_{k+1} \subseteq N_k$ and $\bigcap_{k \in \mathbb{N}} N_k = \{1\}$.*

Proof. Ad 1 $\implies$ 2. Apply the hypothesis to the element $gh^{-1} \in G \setminus \{1\}$ to obtain a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(gh^{-1}) \neq 1$ and equivalently, $\phi(g) \neq \phi(h)$.

Ad 2 $\implies$ 3. Apply the hypothesis to the elements $g, 1 \in G$. We obtain a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq \phi(1) = 1$. Now put $N := \ker \phi$. Then N is a normal subgroup of G, and we have $g \notin N = \ker \phi$ because $\phi(g) \neq 1$. Finally, N is of finite index since the quotient group G/N is isomorphic to F and therefore finite.

Ad 3 $\implies$ 4. Assume that $g \in G \setminus \{1\}$. By assumption, there exists a finite index normal subgroup $N \trianglelefteq G$ such that $g \notin N$. In particular, g is not in the intersection of all finite index subgroups.

Ad 4 $\implies$ 5. By Lemma 1.1, for every finite index subgroup $H \subseteq G$, there exists a finite index normal subgroup $N \trianglelefteq G$ such that $N \subseteq H$. In particular, the intersection of all finite index normal subgroups is contained in the intersection of all finite index subgroups, which is trivial by the hypothesis.

Ad 5 $\implies$ 6. Choose an enumeration $G = \{g_1, g_2, \dots\}$. By the hypothesis, we can choose a finite index normal subgroup $M_j \trianglelefteq G$ such that $g_j \notin M_j$ for all $j \in \mathbb{N}$. For every $k \in \mathbb{N}$, define $N_k := \bigcap_{j=1}^{k} M_j$. Then $(N_k)_k$ is a residual chain: N_k is a finite index normal subgroup of G for each $k \in \mathbb{N}$, because the finite intersection of subgroups preserves finite index and normality. Moreover, we have for every $k \in \mathbb{N}$ that $N_{k+1} = \bigcap_{j=1}^{k+1} M_j = \left(\bigcap_{j=1}^{k} M_j\right) \cap N_k$ and hence $N_{k+1} \subseteq N_k$. In particular, $\bigcap_{k\in\mathbb{N}} N_k = \bigcap_{j\in\mathbb{N}} M_j = \{1\}$.

Ad 6 $\implies$ 1. Consider a residual chain $(N_k)_k$ of G, and let $g \in G \setminus \{1\}$. Since $\bigcap_{k\in\mathbb{N}} N_k = \{1\}$, there exists a $k_0 \in \mathbb{N}$ such that $g \notin N_{k_0}$. Then $G/N_{k_0} \cong F$ is a finite group. Consider the canonical projection map $\phi : G \to G/N_{k_0}$. Then ϕ is a homomorphism to a finite group, such that $\phi(g) \neq 1$. $\square$

Remark 1.4 (set of residual chains)**.** While verifying the first part of Theorem 1.3 is the most accessible way to establish residual finiteness, the existence of residual chains from the sixth part is of great importance. Given a residually finite group G, we will denote the set (G is countable!) of residual chains of G as

$$\mathcal{R}(G) := \{N_* \mid N_* \text{ is a residual chain in } G\}.$$

Notice that some authors impose the stronger requirement that a residually finite group is finitely generated.

Example 1.5 (residually finite groups)**.** In the following cases, we can verify the first part of Theorem 1.3.

- Every finite group G is residually finite. Let $g \in G \setminus \{1\}$. Then the identity map $\mathrm{id}_G : G \to G$ satisfies $\mathrm{id}_G(g) = g \neq 1$.

- The group of integers $\mathbb{Z}$ is residually finite. Let $n \in \mathbb{Z} \setminus \{0\}$. Choose $m := |n| + 1$. Then the canonical projection $\phi : \mathbb{Z} \to \mathbb{Z}/m$ satisfies $\phi(n) = [n] \neq 0$.
- Free groups and free Abelian groups are residually finite (Corollary 1.18).
- The group $\mathrm{GL}_n(\mathbb{Z})$ is residually finite for every $n \geq 1$. Let $M = (M_{ij})_{i,j} \in \mathrm{GL}_n(\mathbb{Z}) \setminus \{I_n\}$. Choose $m := \max\{|M_{ij}| \mid i, j \in \{1, \ldots, n+1\}\}$. Then the reduction homomorphism $\phi : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m)$ satisfies $\phi(M) \neq I_n$.
- A countable nontrivial divisible group G is not residually finite. Recall that a group G is called divisible, if for every $g \in G$ and each $n \geq 1$ there exists $h \in G$ such that $h^n = g$. Examples include the additive groups $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ (only $\mathbb{Q}$ being countable). Every homomorphism ϕ from a divisible group to a finite group F must be trivial: let $\phi : G \to F$ and let $g \in G$. Then there exists an $h \in G$ such that $h^{|F|} = g$. In particular: $\phi(g) = \phi(h^{|F|}) = \phi(h)^{|F|} = 1$.
- Infinite simple groups are not residually finite. In fact, the only finite index normal subgroup of such a group G is G itself. In particular, this violates the third part of Theorem 1.3.
- Let $m, n \in \mathbb{Z}$. The *Baumslag–Solitar group* $\mathrm{BS}(m, n)$ is defined as

  $$\mathrm{BS}(m, n) := \langle a, b \mid ba^m b^{-1} = a^n \rangle.$$

  Meskin showed that $\mathrm{BS}(m, n)$ is residually finite if and only if [Mes72]

  $$|m| = 1, \text{ or } |n| = 1, \text{ or } |m| = |n|.$$

  A group is said to be *Hopfian*, if every surjective endomorphism of G is injective. Baumslag and Solitar showed that $\mathrm{BS}(m, n)$ is Hopfian if and only if m and n have the same prime factors [BS62]. This disproved Higman's claim that every finitely generated one-relator group is Hopfian.
- Every finitely generated residually finite group G is Hopfian [CC10, Theorem 2.4.3]. In particular, the non-Hopfian groups $\mathbb{Q}/\mathbb{Z}$ and $\mathrm{BS}(2, 3)$ are not residually finite. The converse statement is not true: for example, $\mathrm{BS}(2, 4)$ is Hopfian, but not residually finite [CC10, p.76].
- *Surface groups* (see Example 2.17) are residually finite [Hem72].

1.1.1. Index growth of residual chains

One simple but important question which arises is the following: how does the index of a residual chain grow? One needs to distinguish between the case where the group is finite and infinite. This has consequences for the growth of homological invariants (see Example 1.37).

Proposition 1.6 (index growth of residual chains)**.** *Let G be a residually finite group, let $N_* \in \mathcal{R}(G)$ be a residual chain. Then*

1. *If G is finite, then there exists $k_0 \in \mathbb{N}$ such that $[G : N_k] = |G|$ for $k \geq k_0$.*
2. *If G is infinite, then $[G : N_k] \to \infty$ as $k \to \infty$.*

More concisely: $\lim_{k\to\infty}[G : N_k] = |G|$.

Proof. First observe that for all $k \in \mathbb{N}$ we have $N_{k+1} \subseteq N_k \subseteq G$ and hence $[G : N_k] \leq [G : N_{k+1}]$. So $([G : N_k])_{k\in\mathbb{N}}$ is a non-decreasing sequence of natural numbers. If G is finite, then this sequence is bounded by $[G : 1] = |G|$. Hence it must stabilize, i.e., there exists a $k_0 \in \mathbb{N}$ such that

$$N_k = N_{k+1} \quad \forall k \geq k_0.$$

This implies that

$$N_{k_0} = \bigcap_{k=1}^{k_0} N_k = \bigcap_{k\in\mathbb{N}} N_k = \{1\},$$

and it follows that

$$[G : N_k] = [G : 1] = |G| \quad \forall k \geq k_0.$$

If G is infinite, then the sequence of indices cannot be bounded (otherwise the same stabilization argument would force $|G| = [G : 1] < \infty$), so $[G : N_k] \to \infty$ as $k \to \infty$. □

1.1.2. Subgroups

Proposition 1.7 (subgroups)**.** *Every subgroup of a residually finite group is residually finite.*

Proof. Let H be a subgroup of the residually finite group G. Let $h \in H \setminus \{1\}$. In particular, $h \in G \setminus \{1\}$, and therefore there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(h) \neq 1$. The restriction $\phi_{|H} : H \to F$ is a group homomorphism such that $\phi_{|H}(h) \neq 1$. □

Definition 1.8 (cofinal residual chain)**.** Let G be a residually finite group. A residual chain $N_* \in \mathcal{R}(G)$ is called *cofinal* if for every finite index subgroup H, there exists $k_0 \in \mathbb{N}$ such that $H \supseteq N_{k_0}$.

Proposition 1.9 (existence of cofinal chains)**.** *Let G be finitely generated and residually finite. Then there exists a cofinal residual chain N_* in G.*

Proof. For a finitely generated group G, there exist only finitely many finite index normal subgroups of a given index. This follows from the fact that a group homomorphism to a finite group is determined by the images of the generating elements. In particular, the set of *all* finite index normal subgroups of G is countable, say $\{M_j\}_{j\in\mathbb{N}}$. The corresponding residual chain defined as $N_k := \bigcap_{j=0}^{k} M_j$ is cofinal: let $H \subseteq G$ be a finite index subgroup. By Lemma 1.1, there exists a finite index normal subgroup M_{k_0} of G such that $H \supseteq M_{k_0}$. In particular, we have that $H \supseteq M_{k_0} \supseteq M_1 \cap \ldots \cap M_{k_0} = N_{k_0}$. □

Example 1.10 (cofinal chains)**.**

- The residual chain $(2^k\mathbb{Z})_k$ in $\mathbb{Z}$ is *not* cofinal: the finite index subgroup $3\mathbb{Z} \subseteq \mathbb{Z}$ is not contained in $2^k\mathbb{Z}$ for all $k \in \mathbb{N}$.
- The residual chain $((k!)\mathbb{Z})_k$ in $\mathbb{Z}$ is cofinal and arises as the construction in Proposition 1.9 associated to the enumeration of all finite index normal subgroups $\{j\mathbb{Z}\}_{j\in\mathbb{N}}$.
- There exist residually finite groups that *do not* admit cofinal residual chains. Consider the infinitely generated free group $F_{\mathbb{N}}$. It contains uncountably many finite index normal subgroups of index 2. Assume that there exists a cofinal residual chain N_* in $F_{\mathbb{N}}$. Notice that each finite index normal subgroup N can only be contained in finitely many finite index normal subgroups of index 2: given $N_{k_0} \subseteq T \subseteq G$ with $[G : T] = 2$, we have that $[G/N : T/N] = 2$. As G/N is a finite group, there exist only finitely many index 2 subgroups. In particular, only countably many finite index normal subgroups of index 2 can be contained in a residual chain, so there can not exist a cofinal residual chain in $F_{\mathbb{N}}$ as it admits uncountably many such subgroups.

Lemma 1.11 (cofinal chains and subgroups)**.** *Let G be a residually finite group, let $H \subseteq G$ be a finite index subgroup. Let $N_* \in \mathcal{R}(G)$ be cofinal. Then there exists $k_0 \in \mathbb{N}$, such that $(N_k)_{k\geq k_0}$ is a cofinal residual chain in H with*

$$[H : N_k] = [G : H]^{-1} \cdot [G : N_k] \quad \text{for all } k \geq k_0.$$

Proof. Let $H \subseteq G$ be a finite index subgroup. By cofinality of the chain N_*, there exists a $k_0 \in \mathbb{N}$ such that $H \supseteq N_{k_0}$. Clearly, for $k \geq k_0$ the N_k are subgroups of H. The normality follows from the general fact that given a normal subgroup $N_k \trianglelefteq G$ and a subgroup $H \subseteq G$ such that $N_k \subseteq H$ it follows that $N_k \trianglelefteq H$ is a normal subgroup. The chain of subgroups $G \supseteq H \supseteq N_k$ yields that $[G : N_k] = [G : H] \cdot [H : N_k]$. In particular, the index $[H : N_k]$ is finite. □

Caveat 1.12. Due to non-transitivity of normality, there is no converse to the above lemma: a residual chain in H is not neccessarily a residual chain in G.

Lemma 1.13. *Every virtually residually finite group is residually finite.*

Proof. Let G be a group, let H be a finite index subgroup that is residually finite. Note that every finite index subgroup of H is a finite index subgroup of G. By Theorem 1.3.4, we have

$$\bigcap_{N \in \mathcal{F}(G)} N \subseteq \bigcap_{N \in \mathcal{F}(H)} N = \{1\}.$$

In particular, G itself satisfies the fouth part of Theorem 1.3. □

1.1.3. Direct products

Lemma 1.14 (direct products)**.** *Let G and H be groups. Then:*

$$G \times H \text{ is residually finite} \iff G \text{ and } H \text{ are residually finite.}$$

Proof. Ad "⇒". Assume that G is *not* residually finite, witnessed by an element $g \in G \setminus \{1\}$ such that for all pairs of finite groups F and group homomorphisms $\varphi : G \to F$, one has that $\varphi(g) = 1$. Let $\phi : G \times H \to Q$ be a homomorphism to a finite group Q. Then the restriction $\phi_{|G} : G \to Q$ is a group homomorphism, and hence for $g \in G \times H$ we have $\phi(g) = \phi_{|G}(g) = 1$, so $G \times H$ is not residually finite.

Ad "⇐". Let $(g, h) \in G \times H$ be non-trivial, so we may assume that $g \neq 1$. By residual finiteness of G, there exists a finite group F together with a group homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq 1$. In particular, the homomorphism $\varphi \times \mathrm{id}_H$ is a homomorphism to a finite group F and satisfies that $(\varphi \times \mathrm{id}_H)(g, h) = (\varphi(g), h) \neq (1, 1)$. □

Lemma 1.15 (residual chains in direct products)**.** *Let G_1 and G_2 be residually finite groups. Let $N^1_* \in \mathcal{R}(G_1)$ and $N^2_* \in \mathcal{R}(G_2)$. Then $N^1_* \times N^2_* \in \mathcal{R}(G_1 \times G_2)$, and*

$$[G_1 \times G_2 : N^1_k \times N^2_k] = [G_1 : N^1_k] \cdot [G_2 : N^2_k] \quad \text{for all } k \in \mathbb{N}.$$

Proof. Clearly N_* is a nested sequence of normal subgroups of $G_1 \times G_2$, and the isomorphism $(G_1 \times G_2)/(N^1_k \times N^2_k) \cong G_1/N^1_k \times G_2/N^2_k < \infty$ implies the index formula. The trivial intersection property follows from

$$\bigcap_{k\in\mathbb{N}} (N^1_k \times N^2_k) = \Big(\bigcap_{k\in\mathbb{N}} N^1_k\Big) \times \Big(\bigcap_{k\in\mathbb{N}} N^2_k\Big) = \big\{(1, 1)\big\}. \qquad □$$

1.1.4. Free products

Theorem 1.16 (free products)**.** *Let G and H be groups. Then:*

$$G * H \text{ is residually finite} \iff G \text{ and } H \text{ are residually finite.}$$

Proof. Ad "$\Rightarrow$". Assume that G is *not* residually finite, witnessed by an element $g \in G \setminus \{1\}$ such that for all pairs of finite groups F and group homomorphisms $\varphi : G \to F$, one has that $\varphi(g) = 1$. Let $\phi : G * H \to Q$ be a homomorphism to a finite group Q. Then the restriction $\phi_{|G} : G \to Q$ is a group homomorphism, and hence for $g \in G * H$ we have $\phi(g) = \phi_{|G}(g) = 1$.

Ad "$\Leftarrow$". We shall provide a combinatorial proof [Coh89, Prop. 22]. First we show that it suffices to consider finite groups G and H. Let $w \in G * H$ be a non-trivial word. By the standard normal form result, we can uniquely write w as a reduced word $w = s_1 s_2 \cdots s_n$ [Rot09, Theorem 11.52] [Löh17, Outlook 3.3.8], i.e., for all $j \in \{0, \dots, n\}$

- either $s_j \in G \setminus \{1\}$ and $s_{j+1} \in H \setminus \{1\}$,
- or $s_j \in H \setminus \{1\}$ and $s_{j+1} \in G \setminus \{1\}$.

By the residual finiteness of G and H, there exist finite index normal subgroups $N \trianglelefteq G$ and $M \trianglelefteq H$ such that for all $j \in \{0, \dots, n\}$ we have that $s_j \notin N$ if $s_j \in G \setminus \{1\}$ and similarly $s_j \notin M$ if $s_j \in H \setminus \{1\}$. Hence, the image of w under the canonical map $G * H \to (G/N) * (H/M)$ given by $\bar{w} = \bar{s_1}\bar{s_2} \dots \bar{s_n}$ is in reduced form of the same length as w, and so is non-trivial. In particular, it suffices to show residual finiteness of the latter group, which is a free product of finite groups. Assume that G and H are finite. Define the set Ω of all elements of $G*H$ of reduced word length at most n. Since G and H are finite, Ω is a finite set. We define actions of G and H on Ω in the following way: given $g \in G$ and $u \in \Omega$, define $g \cdot u = gu$ if the length of gu after reduction is $\leq 2n$, otherwise $g \cdot u = u$. Similarly for H. This gives us homomorphisms $\varphi_G : G \to \operatorname{Sym}(\Omega)$ and $\varphi_H : H \to \operatorname{Sym}(\Omega)$, which yields a homomorphism $\varphi := \varphi_G * \varphi_H : G * H \to \operatorname{Sym}(\Omega)$ into a finite group. Now, we have that $\varphi(w)(1) = w \cdot 1 = w$, so in particular $\varphi(w) \neq \mathrm{id}_\Omega$ is non-trivial. $\square$

Proposition 1.17 (residual chains in free products)**.** *Let G_1 and G_2 be residually finite groups. There exist set maps*

$$\begin{aligned} \mathcal{R}(G_1 * G_2) &\xrightarrow{\phi} \mathcal{R}(G_1) \times \mathcal{R}(G_2) \\ N_* &\mapsto (N_* \cap G_1, N_* \cap G_2) \\ \mathcal{R}(G_1 * G_2) &\xleftarrow{\psi} \mathcal{R}(G_1) \times \mathcal{R}(G_2) \\ \ker(\pi_*) &\leftarrow\!\shortmid (N_*^1, N_*^2). \end{aligned}$$

such that $\phi \circ \psi = \mathrm{id}_{\mathcal{R}(G_1) \times \mathcal{R}(G_2)}$.

Proof. Ad "ϕ". Given a residual chain $N_* \in \mathcal{R}(G_1 * G_2)$, write $N_k^l := N_k \cap G_l$ for $l \in \{1,2\}$ and $k \in \mathbb{N}$ (where we identify G_l with its image in G). It is clear that N_*^l nested sequence of subgroups in G_l, and by the second isomorphism theorem 1.2.3 every N_k^l is normal in G_l. Moreover, we have

$$\bigcap_{k\in\mathbb{N}} N_k^l = \bigcap_{k\in\mathbb{N}} \big(N_k \cap G_l\big) = \big(\bigcap_{k\in\mathbb{N}} N_k\big) \cap G_l = \{1\} \text{ in } G_l.$$

By Theorem 1.2.4, and the tower of subgroups $G_1 * G_2 \supseteq G_l N_k \supseteq N_k$ we have that

$$[G_l : N_k^l] = [G_l N_k : N_k] = [G_1 * G_2 : N_k] \cdot [G_1 * G_2 : G_l N_k]^{-1} < \infty.$$

Ad "ψ". Let $N_*^l \in \mathcal{R}(G_l)$ for $l \in \{1,2\}$. Define

$$\begin{aligned} \pi_k : G_1 * G_2 &\longrightarrow G_1/N_k^1 \times G_2/N_k^2 \\ G_1 \ni g_1 &\mapsto ([g_1], 1) \\ G_2 \ni g_2 &\mapsto (1, [g_2]) \end{aligned}$$

This induces a well-defined homomorphism by the universal property of free products. Then $M_* := \ker(\pi_*)$ is a residual chain in $G_1 * G_2$ with the property that $M_k \cap G_l = N_k^l$. Clearly, M_* is a sequence of finite index normal subgroups. To see that $M_{k+1} \subseteq M_k$, we observe that the following diagram commutes by the universal property of the free product:

$$\begin{tikzcd} & G_1 * G_2 \arrow[dl, "\pi_{k+1}"'] \arrow[dr, "\pi_k"] & \\ G_1/N_{k+1}^1 \times G_2/N_{k+1}^2 \arrow[rr] & & G_1/N_k^1 \times G_2/N_k^2 \end{tikzcd}$$

where the horizontal map is induced by the inclusions $N_{k+1}^l \subseteq N_k^l$. In particular, we have that $M_{k+1} = \ker(\pi_{k+1}) \subseteq \ker(\pi_k) = M_k$.

Let $1 \neq g \in G_1 * G_2$ be non-trivial, and write g in reduced form alternating between non-trivial letters in G_1 and G_2. Some letter $h \in G_l$ is nontrivial. Since $\bigcap_{k\in\mathbb{N}} N_k^l = \{1\}$, there exists $k \in \mathbb{N}$ with $h \notin N_k^l$. Then $\pi_i(h) \neq 1$, so $\pi_k(g) \neq 1$, hence $g \notin M_k$. Therefore

$$\bigcap_{k\in\mathbb{N}} M_k = \{1\}.$$

Ad $\phi \circ \psi = \mathrm{id}$. Restricting π_k to $G_l \subset G_1 * G_2$ gives for $g \in G_l$

$$\pi_k(g) = \begin{cases} ([g], 1) & \text{if } l = 1, \\ (1, [g]) & \text{if } l = 2. \end{cases}$$

Thus

$$g \in \ker(\pi_k|_{G_l}) \iff [g] = 1 \text{ in } G_l/N_k^l \iff g \in N_k^l,$$

so $M_k \cap G_l = N_k^l$. $\square$

Corollary 1.18 (free groups, Abelian groups)**.** *Finitely generated Abelian groups and finitely generated free groups are residually finite.*

Proof. This follows immediately from the fact that the group of integers and finite groups are residually finite (Example 1.5), that direct products and free products preserve residual finiteness (Theorem 1.14 and Theorem 1.16), as well as the classification of finitely generated Abelian groups (Theorem 2.37). □

1.1.5. Other constructions

We briefly survey less well-behaved inheritance of residual finiteness under other group-theoretic constructions. Determining residual finiteness of amalgamated free products is considerably harder than for free products. In general, the amalgamated product of residually finite groups (over a residually finite common subgroup) is not residually finite.

Theorem 1.19 ([Eva74, Thm 3.1])**.** *Let G be a residually finite group with an element $g \in G$ of infinite order. There exists a residually finite group H together with an element $h \in H$ of infinite order, such that the amalgamated product $G *_{<g>\cong\mathbb{Z}\cong<h>} H$ is not residually finite.*

Corollary 1.20. *The class of residually finite groups G, with the property that for all residually finite groups H we have that $G *_C H$ is residually finite for C a cyclic group, is the exactly the class of residually finite torsion groups.*

Proof. "$\subseteq$" follows from the above theorem: if G is not a torsion group, it contains an element of infinite order and therefore it does not lie in the class. "$\supseteq$" follows from [Bau63, Thm 3]. □

If the amalgamated subgroup is a *retract*, then residual finiteness is preserved.

Definition 1.21 (retract)**.** Let G be a group, let $H \subseteq G$ be a subgroup. H is a *retract of* G, if there exists a surjection $r : G \twoheadrightarrow H$ such that $r_{|H} = \mathrm{id}_H$.

Theorem 1.22 (amalgamation over retract [BE73])**.** *Let G and H be residually finite groups. Consider a group A viewed as a retract subgroup of G and H. Then $G *_A H$ is residually finite.*

For group extensions, a result by Miller serves as a sufficient criterion.

Theorem 1.23 (group extensions [Mil71, p. 29])**.** *Consider a short exact sequence of groups*

$$1 \to N \to G \to Q \to 1,$$

where N and Q are residually finite and N is finitely generated. Assume that one of the following conditions holds:

1. *N has trivial center.*
2. *The sequence splits.*
3. *Q is free or N is non-Abelian free.*

Then G is residually finite.

Remark 1.24. The condition that N is finitely generated is necessary. In general, a (split) extension of a residually finite group by a residually finite group need not to be residually finite. A finitely generated counterexample can be constructed as a certain wreath product [CC10, Remark 2.6.6].

Remark 1.25 (HNN-extensions)**.** In general, HNN-extensions do not preserve residual finiteness. Consider $\mathbb{Z}$ together with the isomorphism $\varphi : 2\mathbb{Z} \to 3\mathbb{Z},\ 2 \mapsto 3$. Then we have

$$\mathbb{Z}*_\varphi \cong \langle a, b \mid ba^2b^{-1} = a^3\rangle \cong \mathrm{BS}(2,3).$$

which is not residually finite (Example 1.5).

Similarly to amalgamated products, under certain technical conditions, HNN-extensions over infinite cyclic subgroups do preserve residual finiteness. The following theorem is due to Kim.

Theorem 1.26 ([Kim04, Theorem 2.5])**.** *Let G be a finitely generated Abelian group. Let $g, h \in G$ be of infinite order. The HNN-extension $G*_{\langle g\rangle \cong \mathbb{Z} \cong \langle h\rangle}$ is residually finite if and only if one of the following holds:*

1. *If A is cyclic, say $A \cong \langle k\rangle$ with $g = k^n$ and $h = k^m$ for $n, m \in \mathbb{Z}$, then $|n| = 1$ or $|m| = 1$ or $|n| = |m|$.*
2. *If A is not cyclic, then $\langle g\rangle \cap \langle h\rangle = \{1\}$ or $g^n = h^{\pm n}$ for some $n \in \mathbb{N}_{\geq 1}$.*

Notice that above criterion generalizes Meskin's characterization of residually finite Baumslag-Solitar groups cited in Example 1.5. A counterexample to the second criterion for $A = \mathbb{Z}^2$ can be found in Kim's article [Kim04, Example 2.6].

1.2. Growth of invariants

The goal of this section is to define an axiomatic notion of "growth" of numerical group invariants on residually finite groups. For this, one considers the asymptotic behavior of the invariant evaluated at subgroups in a residual chain and normalized by the index. A priori, this sequence might not converge and depends on the choice of the residual chain. We observe some generic properties of invariant growth and prove two results on multiplicativity and additivity of invariant growth under free products. We introduce an *absolute* notion of growth, see how it relates to the relative version, and study the *rank gradient*.

Convention 1.27 (extended reals)**.** We write $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ for the set of *extended real numbers*, equipped with the usual order and arithmetic, with the convention that expressions of the indeterminate forms $\infty - \infty$ and $0 \cdot (\pm\infty)$ are undefined. Unless explicitly stated otherwise, all limits and limits superior are taken in $\overline{\mathbb{R}}$.

Proposition 1.28 (properties of limits and limits superior [Rud76, Thm. 3.20])**.** *Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ be sequences in $\overline{\mathbb{R}}$.*

1. Monotonicity. *If $a_k \leq b_k$ for all $k \geq k_0$, then*
$$\lim_{k\to\infty} a_k \leq \lim_{k\to\infty} b_k, \quad \limsup_{k\to\infty} a_k \leq \limsup_{k\to\infty} b_k.$$
2. Addition. *Let $L, S \in \overline{\mathbb{R}}$. Then, provided that the right hand side is not of the form $\infty - \infty$:*
   - *If $a_k \to L$ and $b_k \to S$, then* $\lim(a_k \pm b_k) = S \pm L$;
   - $\limsup(a_k + b_k) \leq \limsup a_k + \limsup b_k$;
   - *If $a_k \to L$, then* $\limsup(a_k + b_k) = L + \limsup b_k$.
3. Multiplication. *Let $L, S \in \overline{\mathbb{R}}$. Then, provided that the right hand side is not of the form $\pm\infty \cdot 0$:*
   - *If $a_k \to L$ and $b_k \to S$, then* $\lim(a_k \cdot b_k) = S \cdot L$;
   - $\limsup(a_k \cdot b_k) \leq (\limsup |a_k|) \cdot (\limsup |b_k|)$;
   - *If $a_k \to L$, then* $\limsup(a_k \cdot b_k) = L \cdot \limsup b_k$;
4. Squeeze to zero. *If* $\limsup a_k = 0$ *and $a_k \geq 0$ for all $k \in \mathbb{N}$, then $a_k \to 0$.*
5. Subsequence criterion. *If $(a_k)_k$ is bounded and $L \in \mathbb{R}$, then $a_k \to L$ if and only if every convergent subsequence has limit L.*

Definition 1.29 (numerical invariant, invariant growth)**.** Let $\mathcal{C}$ be an *is-class of groups*, i.e., a class of groups closed under isomorphisms and passage to finite index subgroups. A *numerical invariant I of groups in $\mathcal{C}$* is a map
$$I : \mathcal{C} \to \overline{\mathbb{R}},$$
which takes the same value for isomorphic groups.

Assume that all groups in $\mathcal{C}$ are residually finite. Let $G \in \mathcal{C}$ and $N_* \in \mathcal{R}(G)$. The *growth of I (or gradient invariant associated to I) with respect to N_** is given by
$$\widehat{I}(G, N_*) := \limsup_{k\to\infty} \frac{I(N_k)}{[G : N_k]} \in \overline{\mathbb{R}}. \tag{1}$$

Remark 1.30 (growth, gradient, generalized L^2-Betti number!?)**.** The study of the growth of invariants in an axiomatic setting as above is rather uncommon. Therefore, there is no standardized terminology. Across the literature, the following tendency seems to have established:

- if the invariant I is viewed topologically, one speaks of *growth* (e.g. homology growth, torsion growth, volume growth);
- if the invariant I is viewed algebraically, one speaks of *gradient* (e.g. rank gradient, deficiency gradient).

To add to the confusion, in view of Lück's approximation theorem (which relates the $\mathbb{Q}$-Betti number growth of groups to L^2-Betti numbers) Avramidi, Okun and Schreve suggest refering to the growth of $\mathbb{F}_p$-Betti numbers as $\mathbb{F}_p$-L^2*-Betti numbers* (eventhough no relation with L^2-Betti numbers is known). In this thesis, we stick to the terms $\mathbb{Q}$- *and* $\mathbb{F}_p$*-Betti number growth*. As one of the goals of this thesis is to employ a unified approach towards growth of invariants, with special emphasis on torsion growth, we choose the term *growth* in the generic setting, but use the term *rank gradient* and *deficiency gradient* to be aligned with the standard literature. We sometimes use the term *homology growth* to refer to both Betti number growth and torsion growth simultaneously.

Convention 1.31 (domain classes for invariant growth)**.** For us, the most important is-classes $\mathcal{C}$ will be:

- the class of all groups.
- the class of all residually finite groups (see Proposition 1.7),
- and the class of finitely generated residually finite groups (see Lemma 1.57).

If not explicitly stated otherwise, we will simply speak of a "(numerical) invariant I" to mean a "numerical invariant I with domain the class of all groups".

Example 1.32 (growth of multiplicative invariant)**.** Assume that I is a numerical invariant which is *multiplicative*, i.e., for every residually finite group G and finite index normal subgroup $H \subseteq G$ one has that $I(H) = [G : H] \cdot I(G)$. Then the invariant growth of I satisfies

$$\widehat{I}(G, N_*) = \limsup_{k\to\infty} \frac{I(N_k)}{[G : N_k]} = \limsup_{k\to\infty} \frac{[G : N_k] \cdot I(G)}{[G : N_k]} = I(G).$$

We can think of the invariant growth of I as being a stabilization of I along finite index normal subgroups. We will see later that it is multiplicative under some conditions.

Question 1.33 (the approximation problem)**.** *In the setting of Definition 1.29:*

1. *Does the sequence in equation* (1) *converge?*
2. *Does* $\widehat{I}(G, N_*)$ *depend on the chain* $N_* \in \mathcal{R}(G)$*?*
3. *Does* $\widehat{I}$ *admit a different interpretation?*

Definition 1.34 (convergence, chain-independence)**.** Let I be a numerical invariant and let G be a residually finite group together with a residual chain $N_* \in \mathcal{R}(G)$. We say that $\widehat{I}(G, N_*)$ is

1. *(given by) a proper limit*, if the sequence (1) defining $\widehat{I}(G, N_*)$ converges;
2. *chain-independent on* G, if for any other residual chain $N'_* \in \mathcal{R}(G)$, we have that
$$\widehat{I}(G, N_*) = \widehat{I}(G, N'_*).$$

We say that $\widehat{I}$ is *(given by) a proper limit*, if the first part holds for all residually finite groups; and that $\widehat{I}$ is *chain-independent*, if the second part holds for all residually finite groups.

Lemma 1.35 (independence of chain implies convergence)**.** *Let I be a numerical invariant and let G be a residually finite group. If $\widehat{I}$ is chain-independent on G, then $\widehat{I}(G, N_*)$ is given by a proper limit for every residual chain $N_* \in \mathcal{R}(G)$.*

Proof. Fix a residual chain $N_* \in \mathcal{R}(G)$, and write $a_k := \frac{I(N_k)}{[G:N_k]}$ for $k \in \mathbb{N}$. Notice that every subsequence of $(a_{k_l})_l$ corresponds to a residual (sub-)chain $(N_{k_l})_l \in \mathcal{R}(G)$, and by chain-independence one has $\limsup_{l\to\infty} a_{k_l} = \limsup_{k\to\infty} a_k$.

Let $\limsup_{k\to\infty} a_k = +\infty$, and suppose that $(a_k)_k$ does not tend to $+\infty$. Then there exists a subsequence $(a_{k_l})_l$ bounded above by some $M \in \mathbb{R}$. By assumption, $\limsup_{l\to\infty} a_{k_l} = +\infty$, which contradicts the choice of $(a_{k_l})_l$. The case $\limsup_{k\to\infty} a_k = -\infty$ is analogous.

If $\limsup_{k\to\infty} a_k = L$ for some $L \in \mathbb{R}$, then $(a_k)_k$ is bounded and by Proposition 1.28.5, it suffices to show that every convergent subsequence of $(a_k)_k$ has limit L. This follows directly from chain-independence and the fact that for a convergent sequence the limit is equal to the limit superior. □

For some classes of groups, one understands finite index normal subgroups particularly well. This allows us to compute growth of invariants quite directly. Let I be an arbitrary numerical invariant.

Example 1.36 (invariant growth of trivial group)**.** Let $G = 1$ be "the" trivial group. Then $\widehat{I}(1) = I(1)$ is constant and a chain-independent proper limit. This holds trivially, as the constant trivial chain is the only residual chain.

Example 1.37 (invariant growth of finite groups)**.** Let G be finite. Then

$$\widehat{I}(G) = \frac{I(1)}{|G|},$$

given by a chain-independent proper limit. This follows from the proof of Proposition 1.6, which shows that every residual chain in G stabilizes at the trivial group, and the index converges to $|G|$.

Example 1.38 (invariant growth of free Abelian groups)**.** Let $m \in \mathbb{N}_{\geq 1}$. Note that every finite index subgroup of $\mathbb{Z}^m$ is itself isomorphic to $\mathbb{Z}^m$. Thus, for every residual chain $N_* \in \mathcal{R}(\mathbb{Z}^m)$:

$$\widehat{I}(\mathbb{Z}^m, N_*) = \limsup_{k\to\infty} \frac{I(N_k)}{[\mathbb{Z}^m : N_k]} = \begin{cases} 0 & \text{if } I(\mathbb{Z}^m) \in \mathbb{R}, \\ \pm\infty & \text{if } I(\mathbb{Z}^m) = \pm\infty. \end{cases}$$

In particular, $\widehat{I}(\mathbb{Z}^m)$ is a chain-independent proper limit.

Example 1.39 (subgroups of free groups)**.** Let $m \in \mathbb{N}_{\geq 1}$ and let F_m be "the" free group of rank m. By the Nielsen–Schreier theorem A.12, every finite index (normal) subgroup H of F_m is a free group of rank $(m-1)\cdot[F_m : H]+1$. In particular, for every residual chain $N_* \in \mathcal{R}(F_m)$ one has

$$\widehat{I}(F_m, N_*) = \limsup_{k\to\infty} \frac{I\big(F_{(m-1)\cdot[F_m:N_k]+1}\big)}{[F_m : N_k]}.$$

This means that the values of I on free groups already determine the invariant growth of a free group. We will use this later to compute torsion growth, Betti number growth and the rank gradient of free groups.

Example 1.40 (subgroups of surface groups)**.** It can be shown that for a *surface group* G associated to a closed surface S of genus g, and a finite index subgroup H of G, H is itself a surface group associated to a closed surface T with Euler characteristic $\chi(T) = [G : H]^{-1} \cdot \chi(S)$. We will use this in Examples 3.7 and 3.8 to compute the Betti number growth of surface groups.

Outlook 1.41 (numerical invariants)**.** Let $n \in \mathbb{N}$. The following numerical invariants will be our main focus in this thesis:

- the rank $d(G) := \min_{\langle S|R\rangle \cong G} |S|$ (Chapter 1.2.4);

- the deficiency $\delta(G) := \min_{\langle S|R\rangle \cong G} |S| - |R|$ (Chapter 1.2.4);
- the n-th $\mathbb{F}$-Betti number for a field $\mathbb{F}$, $b_n(G;\mathbb{F}) := \dim_{\mathbb{F}} H_n(G;\mathbb{F})$ (Chapter 3.1.1);
- the logarithm of the size of the torsion part in integral n-th homology: $\log \operatorname{tors} H_n(G;\mathbb{Z})$ (Chapter 3.1.2).

1.2.1. Absolute growth

Our definition of invariant growth suffers from the (a priori) dependence on the chosen residual chain, and the fact that residual chains are hard to "compare". In the case where the invariant I is well-behaved under passing to subgroups, we can eliminate the dependence on chains and define an *absolute* invariant growth. This notion can be extended to groups that are not residually finite.

Definition 1.42 (submultiplicativity)**.** Let I be a numerical invariant. We say that I is *(subgroup-) submultiplicative*, if for every residually finite group G and every finite index subgroup $H \subseteq G$ one has

$$I(H) \leq I(G) \cdot [G : H]. \tag{2}$$

Definition 1.43 (absolute invariant growth)**.** The *absolute invariant growth of I* of $G \in \mathcal{C}$ is defined as

$$\widehat{I}(G) := \inf_{H \in \mathcal{F}(G)} \frac{I(H)}{[G : H]} \in \overline{\mathbb{R}}. \tag{3}$$

Lemma 1.44 (absolute growth of submultiplicative invariant)**.** *Let I be a submultiplicative numerical invariant, and let G be a residually finite group. Then $\widehat{I}(G, N_*)$ is a proper limit for every residual chain $N_* \in \mathcal{R}(G)$. If I only takes values in $[C, \infty)$ for some $C \in \mathbb{R}$, then this proper limit is finite. Moreover, if G is finitely generated, the absolute growth satisfies*

$$\widehat{I}(G) = \inf_{N_* \in \mathcal{R}(G)} \widehat{I}(G, N_*). \tag{4}$$

In particular: if $\widehat{I}(G, N_*)$ is chain-independent, then for every residual chain $N_* \in \mathcal{R}(G)$

$$\widehat{I}(G) = \widehat{I}(G, N_*).$$

Proof. Let $N_* \in \mathcal{R}(G)$. Submultiplicativity (2) gives for every $k \in \mathbb{N}$

$$\frac{I(N_{k+1})}{[G : N_{k+1}]} \leq \frac{[N_k : N_{k+1}] \cdot I(N_k)}{[N_k : N_{k+1}] \cdot [G : N_k]} = \frac{I(N_k)}{[G : N_k]}.$$

So the sequence defining $\widehat{I}(G, N_*)$ is non-increasing, and thus converges (possibly to $\pm\infty$). If I only takes values in $[C, \infty)$, then the limit is finite.

For every residual chain $N_* \in \mathcal{R}(G)$, we have

$$\widehat{I}(G) = \inf_{H \in \mathcal{F}(G)} \frac{I(H)}{[G:H]} \leq \inf_{k \in \mathbb{N}} \frac{I(N_k)}{[G:N_k]} = \lim_{k \to \infty} \frac{I(N_k)}{[G:N_k]} = \widehat{I}(G, N_*).$$

In particular, $\widehat{I}(G) \leq \inf_{N_* \in \mathcal{R}(G)} \widehat{I}(G, N_*)$.

Since G is finitely generated, Lemma 1.11 implies the existence of a cofinal residual chain $N_* \in \mathcal{R}(G)$. Let $H \in \mathcal{F}(G)$. Then, there exists a $k_0 \in \mathbb{N}$ such that $H \supseteq N_{k_0}$. By submultiplicativity, we have that

$$\frac{I(H)}{[G:H]} \geq \frac{I(N_{k_0})}{[G:N_{k_0}]}.$$

In particular, $\widehat{I}(G) \geq \inf_{N_* \in \mathcal{R}(G)} \widehat{I}(G, N_*)$. $\square$

Convention 1.45 (chain-independence)**.** If $\widehat{I}$ is known to be chain-independent on a group G, we denote this common value as $\widehat{I}(G)$, which is the same notation as for the absolute growth of G. Equality (4) can be seen as a justification for this slight abuse of notation.

Outlook 1.46 (rank gradient)**.** A classical submultiplicative numerical invariant of finitely generated groups is the *rank* (Definition 1.54). This gives rise to the *(absolute) rank gradient*, introduced by Lackenby [Lac05]. By virtue of Lemma 1.44, we will be able to show that the following problem is algorithmically undecidable (Lemma 4.7):

> Given a finitely presented group $G \cong \langle S \mid R \rangle$, is $\mathrm{RG}(G) = 0$?

Remark 1.47 (absolute growth of supermultiplicative invariant)**.** Analogously, we can define *(subgroup-) supermultiplicative* numerical invariants, and the *absolute invariant growth* with sup instead of inf, such that an analogous statement to Lemma 1.44 holds. Since there is no systematic axiomatic framework for growth of invariants in place, the choice in the literature regarding lim sup, lim inf etc. is typically made on a case-by-case basis, depending on the context. For the *deficiency gradient* (Definition 1.67), we will use the sup-variant of absolute growth.

1.2.2. Finite index subgroups

As discussed in the introduction, one of the motivations to study the growth of invariants construction is to force multiplicativity. This is achieved, assuming that the subgroup "fits" into a given residual chain of the ambient group.

Lemma 1.48 (finite index subgroups, relative version)**.** *Let I be a numerical invariant. Let G be a residually finite group together with a residual chain $N_* \in \mathcal{R}(G)$. Let $H \subseteq G$ be a finite index subgroup, such that there exists $k_0 \in \mathbb{N}$ with $N_{k_0} \supseteq H$. Then $(N_k)_{k \geq k_0}$ is a residual chain in H satisfying*

$$\widehat{I}\big(G, (N_k)_k\big) = \frac{1}{[G:H]} \cdot \widehat{I}\big(H, (N_k)_{k \geq k_0}\big).$$

In particular:

- *$\widehat{I}\big(G, (N_k)_k\big)$ is given by a proper limit if and only if $\widehat{I}\big(H, (N_k)_{k \geq k_0}\big)$ is;*
- *if $\widehat{I}(G)$ and $\widehat{I}(H)$ are chain-independent, then we have*

$$\widehat{I}(G) = \frac{1}{[G:H]} \cdot \widehat{I}(H).$$

Proof. One easily checks that $(N_k)_{k \geq k_0}$ is a residual chain in H. Then

$$\begin{aligned}
\widehat{I}\big(G, (N_k)_k\big) &= \limsup_{k \to \infty} \frac{I(N_k)}{[G:N_k]} \\
&= \limsup_{k \geq k_0, k \to \infty} \frac{I(N_k)}{[G:H][H:N_k]} \\
&= \frac{1}{[G:H]} \cdot \limsup_{k \geq k_0, k \to \infty} \frac{I(N_k)}{[H:N_k]} \\
&= \frac{1}{[G:H]} \cdot \widehat{I}\big(H, (N_k)_{k \geq k_0}\big).
\end{aligned}$$

□

Notice that in the case where N_* is a cofinal residual chain, the condition on H is satisfied automatically for any finite index subgroup H of G.

Example 1.49 (direct product with finite group)**.** Let A be a residually finite group, and let $N_* \in \mathcal{R}(A)$ be a residual chain. Let $M \in \mathbb{N}_{\geq 2}$ and define $G := A \times \mathbb{Z}/M$. Then G is residually finite (Lemma 1.14) and $N_* \times 1$ is a residual chain in G (Lemma 1.15). Notice that G contains $A \times 1$ ($\cong A$) as a finite index subgroup of index M, and that $N_* \times 1$ is contained in $A \times 1$. Let I be a numerical invariant. By Lemma 1.48, we get that

$$\widehat{I}(G, N_* \times 1) = \frac{1}{M} \cdot \widehat{I}(A \times 1, N_* \times 1) = \frac{1}{M} \cdot \widehat{I}(A, N_*). \tag{5}$$

In the second equality we have used that I is invariant under isomorphisms and that $[A \times 1 : N \times 1] = [A : N]$. We will use this construction later to obtain more refined realizability results for growth of homological invariants (Chapter 4.2).

Lemma 1.50 (finite index subgroups, absolute version)**.** *Assume that I is a submultiplicative numerical invariant. Let G be a group, let H be a finite index subgroup. Then*

$$\widehat{I}(G) = \frac{1}{[G:H]} \cdot \widehat{I}(H).$$

Proof. Let $H \subseteq G$ be a finite index subgroup. We have

$$\begin{aligned}
\widehat{I}(G) &= \inf_{K \in \mathcal{F}(G)} \frac{I(K)}{[G:K]} && \text{(definition)}\\
&= \inf_{K \in \mathcal{F}(G)} \frac{I(K \cap H)}{[G:K \cap H]} && \text{(submultiplicativity)}\\
&= \inf_{K \in \mathcal{F}(G)} \frac{I(K \cap H)}{[G:H] \cdot [H:K \cap H]} && \text{(tower } K \cap H \subseteq H \subseteq G)\\
&= \frac{1}{[G:H]} \cdot \inf_{L \in \mathcal{F}(H)} \frac{I(L)}{[H:L]} && (K \cap H \leftrightarrow L)\\
&= \frac{1}{[G:H]} \cdot \widehat{I}(H). && \text{(definition)}
\end{aligned}$$

By submultiplicativity, for every $K \in \mathcal{F}(G)$ we have $\frac{I(K \cap H)}{[G:K \cap H]} \leq \frac{I(K)}{[G:K]}$. In particular, we get "$\geq$" in the second equality. "$\leq$" is clear, because $K \cap H \in \mathcal{F}(G)$. For the fourth equality: to $K \in \mathcal{F}(G)$, assign $K \cap H \in \mathcal{F}(H)$ such that $K \cap H \subseteq K$. Conversely, every $L \in \mathcal{F}(H)$ is of the form $L = K \cap H$ where $K := L \in \mathcal{F}(G)$. □

1.2.3. Free products

The three main numerical invariants in this thesis are all additive under free products of groups. In this section we show that this property is passed onto the invariant growth. The main input comes from Bass–Serre theory, which allows to decompose finite index normal subgroups of free products of groups.

Lemma 1.51 (subgroups of free products)**.** *Let G_1 and G_2 be groups. Let $N \trianglelefteq G_1 * G_2$ be a finite index normal subgroup of index $d := [G_1 * G_2 : N]$. For $l \in \{1,2\}$, we have that $N_l := N \cap G_l$ are finite index normal subgroups in G_l, say of index $d_l := [G_l : N_l]$. Then, N has the following decomposition:*

$$N \cong \underbrace{N_1 * \ldots * N_1}_{\frac{d}{d_1}-many} * \underbrace{N_2 * \ldots * N_2}_{\frac{d}{d_2}-many} * F_r = (N_1)^{*(d/d_1)} * (N_2)^{*(d/d_2)} * F_r,$$

where F is a free group of rank $r = d - \frac{d}{d_1} - \frac{d}{d_2} + 1$.

Proof. The proof uses techniques of Bass–Serre theory and can be found in the appendix section A. □

Theorem 1.52 (invariant growth of free products)**.** *Let I be a numerical invariant with values in $[C, \infty]$ for $C \in \mathbb{R}$, such that for all residually finite groups G_1 and G_2*

$$I(G_1 * G_2) = I(G_1) + I(G_2).$$

Then for every residual chain $N_* \in \mathcal{R}(G)$ of the free product $G := G_1 * G_2$, with associated residual chains $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ (Proposition 1.17), we have for all $k \in \mathbb{N}$:

$$\frac{I(N_k)}{[G : N_k]} = \frac{I(N_k^1)}{[G_1 : N_k^1]} + \frac{I(N_k^2)}{[G_2 : N_k^2]} + I(\mathbb{Z}) \cdot \frac{r_k}{[G : N_k]},$$

where

$$r_k := [G : N_k] - \frac{[G : N_k]}{[G_1 : N_k^1]} - \frac{[G : N_k]}{[G_2 : N_k^2]} + 1.$$

The associated invariant growth satisfies

$$\widehat{I}(G, N_*) \leq \widehat{I}(G_1, N_*^1) + \widehat{I}(G_2, N_*^2) + I(\mathbb{Z}) \cdot \Big(1 - \frac{1}{|G_1|} - \frac{1}{|G_2|} + \frac{1}{|G|}\Big), \qquad (6)$$

with equality if $\widehat{I}(G_1, N_*^1)$ is given by a proper limit. In particular, $\widehat{I}(G_1 * G_2)$ is chain-independent if $\widehat{I}(G_1)$ and $\widehat{I}(G_2)$ are chain-independent.

Proof. For every $k \in \mathbb{N}$ and $l \in \{1, 2\}$, denote $d_k = [G : N_k]$ and $d_k^l := [G_l : N_k^l]$. By the Bass–Serre decomposition in Lemma 1.51, we have for all $k \in \mathbb{N}$

$$N_k \cong (N_k^1)^{*(d_k/d_k^1)} * (N_k^2)^{*(d_k/d_k^2)} * F_{r_k},$$

where r_k is defined as above. Therefore we have

$$\begin{aligned}
\frac{I(N_k)}{d_k} &= \frac{1}{d_k} \cdot I\Big((N_k^1)^{*(d_k/d_k^1)} * (N_k^2)^{*(d_k/d_k^2)} * \mathbb{Z}^{*r_k}\Big) \\
&= \frac{1}{d_k} \cdot \Big(\frac{d_k}{d_k^1} \cdot I(N_k^1) + \frac{d_k}{d_k^2} \cdot I(N_k^2) + r_k \cdot I(\mathbb{Z})\Big) \\
&= \frac{I(N_k^1)}{d_k^1} + \frac{I(N_k^2)}{d_k^2} + I(\mathbb{Z}) \cdot \frac{r_k}{d_k},
\end{aligned}$$

where we have used the free product additivity of I in the second equality. The claim follows from passing to the limit superior, using that none of the involved terms has limit superior equal to $-\infty$ by assumption on I. □

Corollary 1.53 (special cases)**.** *Consider I, G_1, G_2, N_*, N_*^1 and N_*^2 as in Theorem 1.52. Assume that*

1. *$I(\mathbb{Z}) = 0$. Then:*

$$\widehat{I}(G_1 * G_2, N_*) \leq \widehat{I}(G_1, N_*^1) + \widehat{I}(G_2, N_*^2).$$

2. *$I(\mathbb{Z}) = 1$. Then:*

$$\widehat{I}(G_1 * G_2, N_*) \leq \widehat{I}(G_1, N_*^1) + \widehat{I}(G_2, N_*^2) + 1 - \frac{1}{|G_1|} - \frac{1}{|G_2|} + \frac{1}{|G_1 * G_2|}.$$

In both cases, one has equality if $\widehat{I}(G_1, N_*^1)$ is given by a proper limit.

1.2.4. Combinatorial invariants

Before delving into homological group invariants, we introduce the *rank gradient*, a notion due to Lackenby [Lac05]. It is submultiplicative for finitely generated groups, and therefore admits a notion of absolute growth. It is related to Gaboriau's *cost of a group* and the first L^2-Betti number. For more details, we refer to the article by Abért–Gelander–Nikolov [AGN17] and Abért–Nikolov [AN12].

Definition 1.54 (rank of a group)**.** Let G be a group. Define its *rank* $d(G)$ to be the minimal number of generators of G:

$$d(G) := \min \bigl\{ |S| \mid S \subseteq G \text{ generates } G \bigr\} \in \mathbb{N} \cup \{\infty\}.$$

Example 1.55 (ranks of groups)**.** Let G be a group and $m \in \mathbb{N}_{\geq 1}$.

- $d(G) = 0$ if and only if $G \cong 1$ is trivial.
- By definition, $d(G) = 1$ if and only if G is cyclic.
- By definition, $d(G) < \infty$ if and only if G is finitely generated.
- $d(\mathbb{Z}^m) = m$. Viewing the Abelian group $\mathbb{Z}^m$ as a $\mathbb{Z}$-module, we have that a (group-theoretic) generating set correponds to a (module-theoretic) $\mathbb{Z}$-basis. One easily checks that that $\operatorname{rk}_{\mathbb{Z}} \mathbb{Z}^m = m$.
- $d(F_m) = m$. Here, "$\leq$" is witnessed by the free generating set. The image of this generating set in $\mathbb{Z}^m$ under the surjective Abelianization map is a generating set of $\mathbb{Z}^m$. By the previous example, the generating set must consist of at least m elements.
- $d\bigl(\pi_1(\Sigma_g)\bigr) = 2g$, where $\pi_1(\Sigma_g)$ is an orientable-surface group of genus g introduced in Example 2.17.

Theorem 1.56 (Grushko–Neumann theorem)**.** *Let G_1 and G_2 be finitely generated groups. Then*

$$d(G_1 * G_2) = d(G_1) + d(G_2).$$

Proof. "$\leq$" is clear by considering the canonical presentation of the free product. The other direction requires work and can be proved by Bass–Serre theory [Chi76]. □

Lemma 1.57 (submultiplicativity of the rank)**.** *Let G be a finitely generated group. Then for every finite index subgroup H of G*

$$d(H) - 1 \leq \big(d(G) - 1\big) \cdot [G : H]. \tag{7}$$

If G is free, the above is an equality. In particular: both $d(-)$ and $d(-) - 1$ are submultiplicative invariants of finitely generated residually finite groups.

Proof. If G is free, then equality in equation (7) holds by the Nielsen–Schreier theorem, which is proved in the appendix (Theorem A.12). Suppose that G is finitely generated. Then there exists a presentation of G with finitely many generators, in particular we can express G as a quotient of a free group F of finite rank $d(F)$ by a normal subgroup. Let $\pi : F \twoheadrightarrow G$ be the projection map. If $H \subseteq G$ is a finite index subgroup, then $\pi^{-1}(H)$ is a finite index subgroup of F. By the Nielsen–Schreier theorem, $\pi^{-1}(H)$ is isomorphic to a free group of rank $\big(d(G) - 1\big) \cdot [G : H] + 1$. As π is surjective, the image of a generating set is a generating set, so

$$d(H) \leq d\big(\pi^{-1}(H)\big) = \big(d(G) - 1\big) \cdot [G : H] + 1.$$ □

Definition 1.58 ((relative) rank gradient)**.** Let G be a residually finite group and let $N_* \in \mathcal{R}(G)$ be a residual chain. Then the *rank gradient of G with respect to N_** is given by

$$\mathrm{RG}(G, N_*) := \limsup_{k\to\infty} \frac{d(N_k) - 1}{[G : N_k]} = \widehat{d}(G, N_*) - \frac{1}{|G|}.$$

Remark 1.59 (digesting the definition)**.** By Lemma 1.57, $d(-) - 1$ is a submultiplicative invariant of finitely generated groups. Therefore, $\mathrm{RG}(G, N_*)$ is a proper limit for a finitely generated group G (Lemma 1.44). There are several reasons to define the rank gradient of G with this "shift" $-|G|^{-1}$. One advantage is that we get a nice formula for computing the rank gradient of free products (Corollary 1.61).

Example 1.60 (rank gradients)**.**

- By Example 1.37, if G is finite, then for every residual chain $N_* \in \mathcal{R}(G)$
$$\mathrm{RG}(G, N_*) = -\frac{1}{|G|}.$$
Conversely, if $\mathrm{RG}(G, N_*) < 0$, then G is finite: there must exist $k \in \mathbb{N}$ such that $d(N_k) - 1 < 0$, but this implies that $N_k \cong 1$ is trivial, and thus G is finite.

- By Example 1.38, for a finitely generated free Abelian group $G = \mathbb{Z}^m$ with $m \in \mathbb{N}_{\geq 1}$ one has for every residual chain $N_* \in \mathcal{R}(G)$
$$\mathrm{RG}(G, N_*) = 0.$$

- By Example 1.39, for a finitely generated free group $G = F_m$ with $m \in \mathbb{N}_{\geq 1}$ one has for every residual chain $N_* \in \mathcal{R}(G)$
$$\mathrm{RG}(G, N_*) = \lim_{k\to\infty} \frac{d\big(F_{(m-1)\cdot[G:N_k]+1}\big) - 1}{[G : N_k]} = \lim_{k\to\infty} (m-1) + \frac{1}{[G : N_k]} = m - 1.$$

Corollary 1.61 (rank gradient of free products)**.** *Let G_1 and G_2 be finitely generated residually finite groups. Let $N_* \in \mathcal{R}(G_1 * G_2)$ be a residual chain and let $N_*^l \in \mathcal{R}(G_l)$ be the associated residual chains constructed as in Proposition 1.17. Then*
$$\mathrm{RG}(G_1 * G_2, N_*) = \mathrm{RG}(G_1, N_*^1) + \mathrm{RG}(G_2, N_*^2) + 1.$$

Proof. By the Grushko–Neumann theorem 1.56, the rank is additive under free products and we have that $d(\mathbb{Z}) = 1$ by Example 1.55. We are thus in the setting of Corollary 1.53.2, which yields
$$\widehat{d}(G_1 * G_2, N_*) = \widehat{d}(G_1, N_*^1) + \widehat{d}(G_2, N_*^2) + 1 - \frac{1}{|G_1|} - \frac{1}{|G_2|} + \frac{1}{|G_1 * G_2|}.$$
Rearranging the terms gives the desired equality
$$\widehat{d}(G_1 * G_2, N_*) - \frac{1}{|G_1 * G_2|} = \left(\widehat{d}(G_1, N_*^1) - \frac{1}{|G_1|}\right) + \left(\widehat{d}(G_2, N_*^2) - \frac{1}{|G_2|}\right) + 1. \quad \square$$

Remark 1.62. The proof of Corollary 1.61 is in this form is due to Abért, Jaikin and Nikolov [AJN11, Proposition 8]. It served as the original motivation for this thesis to investigate the behavior of invariant growth under free products axiomatically.

Definition 1.63 (absolute rank gradient)**.** Let G be a (non-necessarily residually finite) group. We define its *absolute rank gradient* as the absolute invariant growth of $d(-)-1$, i.e., as

$$\mathrm{RG}(G) := \inf_{H \in \mathcal{F}(G)} \frac{d(H)-1}{[G:H]}.$$

Example 1.64 (negative absolute rank gradient)**.** Let us characterize the groups with negative rank gradient. Assume that G is a group such that $\mathrm{RG}(G) < 0$. Therefore, there exists a finite index subgroup H of G such that $d(H) - 1 < 0$. By Example 1.55, we get that $H \cong 1$ is trivial. So G is a finite group of cardinality $[G : H]$. The infimum in the definition of $\mathrm{RG}(G)$ becomes the minimum $-\frac{1}{|G|}$, achieved by the trivial subgroup. In particular, the minimal possible rank gradient is $-\frac{1}{2}$, realized by the group $\mathbb{Z}/2$.

Using the same Bass–Serre decomposition, we recover a result from Lackenby's original paper [Lac05, Proposition 3.2]:

Corollary 1.65. *Let $G = G_1 * G_2$ be the free product of two non-trivial finitely generated groups. Then $\mathrm{RG}(G) > 0$ if and only if G_1 and G_2 are not both isomorphic to $\mathbb{Z}/2$.*

Proof. The "only if" part is clear: $\mathbb{Z}/2 * \mathbb{Z}/2$ has vanishing rank gradient, which easily follows from additivity under free products, or the fact that $\mathbb{Z}/2 * \mathbb{Z}/2 \cong D_\infty$ contains $\mathbb{Z}$ as a finite index subgroup. Now consider an arbitrary finite index normal subgroup $N \subseteq G$. Write $N_l := N \cap G_l$ for $l \in \{1, 2\}$. Then by Lemma 1.51 we have a decomposition

$$N \cong \underbrace{N_1 * \ldots * N_1}_{\frac{[G:N]}{[G_1:N_1]}-\text{many}} * \underbrace{N_2 * \ldots * N_2}_{\frac{[G:N]}{[G_2:N_2]}-\text{many}} * F_r,$$

where $r = 1 + [G : N] - [G_1 : N_1]^{-1} - [G_2 : N_2]^{-1}$. Using the Grushko–Neumann theorem and $d(F_r) = r$ we obtain

$$\begin{aligned}
\frac{d(N)-1}{[G:N]} &= \frac{d(N_1)-1}{[G_1:N_1]} + \frac{d(N_2)-1}{[G_2:N_2]} + 1 && \text{(Theorem 1.56)} \\
&\geq 1 - \frac{1}{[G_1:N_1]} - \frac{1}{[G_2:N_2]} && \big(d(N_l) \geq 0\big) \\
&\geq \frac{1}{6}. && \text{(Claim)}
\end{aligned}$$

If the claimed lower bound holds, then we are done by the definition of the absolute rank gradient and the fact that N was arbitrary. Now we check the possibilities for the denominators:

- If $[G_1 : N_1] = [G_2 : N_2] = 1$, then $N_1 \cong G_1$ and $N_2 \cong G_2$ and we have that

$$\frac{d(N)-1}{[G:N]} = \frac{d(G_1)-1}{1} + \frac{d(G_2)-1}{1} + 1 \geq 0 + 0 + 1 = 1,$$

since G_1 and G_2 are both non-trivial.

- If $[G_1 : N_1] = 1$ and $[G_2 : N_2] \geq 2$, then $N_1 \cong G_1$ and the above equation becomes

$$\frac{d(N)-1}{[G:N]} = d(G_1) + \frac{d(N_2)-1}{[G_2:N_2]} \geq 1 - \frac{1}{2} = \frac{1}{2},$$

because $d(G_1) \geq 1$ and $d(N_2) \geq 0$.

- If $[G_1 : N_1] = [G_2 : N_2] = 2$, then at least one of N_1 and N_2 must be non-trivial: otherwise $N_1 \cong 1 \cong N_2$ would imply that $G_1 \cong \mathbb{Z}/2 \cong G_2$. So assume that $N_1 \ncong 1$, in particular

$$\frac{d(N)-1}{[G:N]} = \frac{d(N_1)-1}{2} + \frac{d(N_2)-1}{2} + 1 \geq 0 - \frac{1}{2} + 1 = \frac{1}{2}.$$

- Otherwise, $[G_1 : N_1] \geq 2$ and $[G_2 : N_2] \geq 3$ and the inequality gives

$$\frac{d(N)-1}{[G:N]} \geq 1 - \frac{1}{[G_1:N_1]} - \frac{1}{[G_2:N_2]} \geq \frac{1}{6}.$$

In all cases, $\frac{d(N)-1}{[G:N]} \geq \frac{1}{6}$ independently of N, hence $\mathrm{RG}(G) \geq \frac{1}{6} > 0$. □

Example 1.66 (absolute rank gradient of free group)**.** Let H be a finite index subgroup of a finitely generated free group F_m. Then, by the Nielsen–Schreier theorem one has

$$d(H) - 1 = \big(d(F_m) - 1\big) \cdot [F_m : H],$$

and so in particular

$$\mathrm{RG}(F_m) = \inf_{H \in \mathcal{F}(F_m)} \frac{d(H)-1}{[F_m:H]} = d(F_m) - 1 = m - 1.$$

A higher dimensional analogue of the rank gradient is given by the *deficiency gradient*. Eventhough many results about the deficiency (gradient) involve group homology and classifying spaces, we choose to introduce this a priori purely algebraically defined invariant in this section.

Definition 1.67 (deficiency, deficiency gradient)**.** Let G be a finitely presented group. The *deficiency* $\delta(G)$ *of* G is defined as

$$\delta(G) := \max \left\{ |S| - |R| \;\middle|\; G \cong \langle S \mid R \rangle \right\} \in \mathbb{Z}.$$

The invariant growth of $\delta(-) - 1$ is called the *deficiency gradient* DG: given a finitely presented residually finite group G and a residual chain $N_* \in \mathcal{R}(G)$ we define

$$\mathrm{DG}(G, N_*) := \limsup_{k \to \infty} \frac{\delta(N_k) - 1}{[G : N_k]}.$$

We also define the *absolute deficiency gradient* of a finitely presented group as

$$\mathrm{DG}(G) := \sup_{H \in \mathcal{F}(G)} \frac{\delta(G) - 1}{[G : H]}.$$

Example 1.68 (deficiency)**.**

- If G is "the" trivial group, then $\delta(G) = 0 - 0$, realized by the empty presentation $\langle \mid \rangle$.
- Let G be finite. Then $\delta(G) \leq 0$. Assume that G has deficiency ≥ 1, witnessed by a finite presentation $\langle S \mid R \rangle$ of G such that $|S| > |R|$. This implies that G has infinite abelianization, thus it cannot be finite. Gardam showed that every negative integer arises as the deficiency of a finite group [Gar21].

Lemma 1.69 (properties of deficiency gradient)**.** *Let G be a finitely presented residually finite group, and let $N_* \in \mathcal{R}(G)$ be a residual chain. Let $H \subseteq G$ be a finite index subgroup. Then*

1. $\delta(G) \leq d(G)$,
2. $\delta(G) - 1 \leq \frac{\delta(H)-1}{[G:H]} \leq \frac{d(H)-1}{[G:H]} \leq d(G) - 1$,
3. $\mathrm{DG}(G, N_*)$ *is a proper limit,*
4. $\mathrm{DG}(G, N_*) \leq \mathrm{RG}(G, N_*)$.

Proof. A proof can be found in the article by Kar and Nikolov [KN16]. The third and fourth part follow immediately from the first two. □

Example 1.70 (positive deficiency gradient)**.** Given $n, m \in \mathbb{N}$, $\mathrm{DG}(F_n \times F_m) = -(n-1)(m-1)$, while the deficiency gradient of a torsion-free one relator group defined on d generators is $d - 2$ [KN16].

Caveat 1.71 (free products)**.** The deficiency of a group is *not* additive under free products: a counterexample is $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ [HLM].

In light of this, we cannot apply Corollary 1.52. Kar and Nikolov pose the following question:

Question 1.72 (deficiency gradient of free products [KN16, Question 14])**.** *Let G_1 and G_2 be (torsion-free) residually finite groups and let $N_* \in \mathcal{R}(G_1 * G_2)$ be a residual chain. Do we have that*

$$\mathrm{DG}(G_1 * G_2, N_*) = \mathrm{DG}(G_1, N_*^1) + \mathrm{DG}(G_2, N_*^2) + 1?$$

Here, N_*^1 and N_*^2 are defined as in Proposition 1.17.

In the question above, walking through the proof of Corollary 1.52, one can conclude "$\geq$".

2. Homological invariants of groups

In the previous chapter, we have set up a machine that captures the growth of numerical invariants defined on residually finite groups. In this chapter, we will introduce the *homological invariants* of groups alluded to in the introduction. In Chapter 2.1, we review the construction of group homology, explore the topological viewpoint and introduce *finiteness properties of groups*. In Chapter 2.2, we extract the homological invariants from group homology and compute them in some examples. We then state the universal coefficient theorem and use it to study the relationship between Betti numbers with different field coefficients, and torsion growth. Finally, we prove that in the presence of a suitable finiteness condition, the homological invariants stay controlled linearly in the index under passing to finite index subgroups. The standard reference for group (co)homology is Brown's book [Bro82].

Overview of this chapter.

2.1. Group homology

Group (co)homology is an extremely rich mathematical theory, which can be viewed from many different perspectives. Group homology of a group G is defined as the homology of the coinvariants of a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over the group ring $\mathbb{Z}G$. Given a path-connected CW-complex X with contractible universal covering $\widetilde{X}$, one canonical choice of a projective resolution associated to its fundamental group $G = \pi_1(X)$ is the singular/cellular chain complex of $\widetilde{X}$. It turns out one can assign such a CW-complex X to every group G and use the singular (or cellular) homology of X to compute the group homology of $G = \pi_1(X)$, leading to the notion of a *classifying space*. The exposition in this chapter is inspired by Löh's lecture notes [Löh19].

2.1.1. Group rings

Definition 2.1 (group ring)**.** Let G be a group. The *group ring of* G is the unital ring $\mathbb{Z}G$ with underlying Abelian group

$$\bigoplus_{g\in G} \mathbb{Z} = \left\{ \sum_{g\in G} a_g \cdot g \;\middle|\; a_g \in \mathbb{Z},\ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

and with multiplication given by

$$\Big(\sum_{g\in G} a_g \cdot g\Big) \cdot \Big(\sum_{g\in G} b_g \cdot g\Big) := \sum_{g\in G}\sum_{h\in G} a_h \cdot b_{h^{-1}g} \cdot g.$$

The unit is given by $1 \cdot e$.

Remark 2.2 ($\mathbb{Z}G$-modules and G-actions)**.** Let G be a group.

- The data of a (left) $\mathbb{Z}G$-module A is precisely the data of an Abelian group A together with a (left) G-action by group automorphisms. In particular, every Abelian group A gives rise to "the" *trivial* $\mathbb{Z}G$*-module* A defined as the $\mathbb{Z}G$-module associated to A with respect to the trivial action of G on A.

- $\mathbb{Z}G$ is commutative if and only if G is Abelian. We use the following convention: given two (left) $\mathbb{Z}G$-modules A and B, we define the $\mathbb{Z}$-module

$$A \otimes_{\mathbb{Z}G} B := \operatorname{Inv}(A) \otimes_{\mathbb{Z}G} B,$$

where $\operatorname{Inv}(A)$ is the right $\mathbb{Z}G$-module with same underlying Abelian group A and right G-action

$$\begin{aligned} A \times G &\to A \\ (a, g) &\mapsto g^{-1} \cdot a. \end{aligned}$$

- Let $n \in \mathbb{N}_{\geq 2}$. The following are basic examples of group rings [Bro82, Examples I.2]:

$$\begin{aligned}
\mathbb{Z}[1] &\cong_{\text{Ring}} \mathbb{Z};\\
\mathbb{Z}[\mathbb{Z}] &\cong_{\text{Ring}} \mathbb{Z}[X, X^{-1}];\\
\mathbb{Z}[\mathbb{Z}/n] &\cong_{\text{Ring}} \mathbb{Z}[X]/(X^n - 1).
\end{aligned}$$

- In general, the ring structure of group rings is rather mysterious. One version of the open *Kaplansky conjecture* asks whether the group ring $\mathbb{Z}G$ is a domain for G a torsion-free group.

- Given a G-set S, one obtains a $\mathbb{Z}G$-module $\mathbb{Z}[S]$ with underlying Abelian group
$$\mathbb{Z}[S] := \bigoplus_{s \in S} \mathbb{Z},$$
and with $\mathbb{Z}G$-module structure given by
$$\Big(\sum_{g \in G} a_g \cdot g\Big) \cdot \Big(\sum_{s \in S} b_s \cdot s\Big) := \sum_{s \in S} \sum_{g \in G} (a_g \cdot b_s) \cdot (g \cdot s).$$
If the action of G on S is free, then $\mathbb{Z}[S]$ is a free $\mathbb{Z}G$-module:
$$S = \coprod_{[s] \in S \setminus G} G \cdot s \quad \Longrightarrow \quad \mathbb{Z}[S] \cong_{\mathbb{Z}G} \bigoplus_{[s] \in S \setminus G} \mathbb{Z}G.$$

2.1.2. Projective resolutions

Let R be a ring and M an R-module. We say that M is *projective over* R if it is a direct summand of a free R-module (in particular, free R-modules are projective). A *projective resolution of* M *over* R consists of an R-chain complex (P_*, ∂_*) of projective R-modules and an R-homomorphism ε called *augmentation map* such that the following sequence is exact:

$$\cdots \to P_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0.$$

We use the concise notation (P_*, ε) to denote this sequence, leaving the boundary homomorphisms implicit.

Theorem 2.3 (existence and uniqueness of projective resolutions)**.** *Let R be a ring and let M be an R-module. Then M admits a projective resolution over R, which is unique up to R-chain homotopy equivalence.*

Proof. A proof can be found in Weibel's book [Wei94, Lemma 2.2.5, Theorem 2.2.6]. □

Group homology is defined as the chain homology of the coinvariants of a projective resolution of the trivial $\mathbb{Z}$-module over $\mathbb{Z}G$.

Convention 2.4. We always regard $\mathbb{Z}$ as "the" trivial $\mathbb{Z}G$-module.

Example 2.5 (simplicial resolution)**.** Let G be a group. One constructs a projective resolution $\big(C_*(G), \varepsilon\big)$ of $\mathbb{Z}$ over $\mathbb{Z}G$ as follows:

- For $k \in \mathbb{N}$ define
$$C_k(G) := \bigoplus_{G^{k+1}} \mathbb{Z}$$
with G-action induced by the diagonal action
$$\begin{aligned} G \times G^{k+1} &\to G^{k+1} \\ \big(g, (g_0, \dots, g_k)\big) &\mapsto (g \cdot g_0, \dots, g \cdot g_k). \end{aligned}$$

- For all $k \in \mathbb{N}_{\geq 1}$ define the boundary operator as the $\mathbb{Z}$-linear map with
$$\begin{aligned} \partial_k : C_k(G) &\to C_{k-1}(G) \\ G^{k+1} \ni (g_0, \dots, g_k) &\mapsto \sum_{j=0}^{k} (-1)^j \cdot (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_k). \end{aligned}$$
We set $\partial_0 := 0 : C_0(G) \to 0$.

- The augmentation map is induced by
$$\begin{aligned} \varepsilon : C_0(G) = \mathbb{Z}G &\to \mathbb{Z} \\ G \ni g &\mapsto 1. \end{aligned}$$

It is a standard exercise to check that this is indeed a projective resolution, called the *simplicial resolution* [Bro82, Chapter V.5].

Example 2.6 (taking coinvariants of a projective resolution of $\mathbb{Z}$)**.** Let (P_*, ε) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Given a $\mathbb{Z}G$-module A, we obtain a $\mathbb{Z}$-chain complex $\big(P_* \otimes_{\mathbb{Z}G} A, \varepsilon \otimes_{\mathbb{Z}G} A\big)$ given by

$$\dots \xrightarrow{\partial_2 \otimes_{\mathbb{Z}G} \mathrm{id}_A} P_1 \otimes_{\mathbb{Z}G} A \xrightarrow{\partial_1 \otimes_{\mathbb{Z}G} \mathrm{id}_A} P_0 \otimes_{\mathbb{Z}G} A \xrightarrow{\varepsilon \otimes_{\mathbb{Z}G} \mathrm{id}_A} \mathbb{Z} \otimes_{\mathbb{Z}G} A \to 0. \tag{8}$$

This is functorial: given a $\mathbb{Z}G$-chain map $f_*(P_*, \varepsilon) \to (P'_*, \varepsilon')$, then the sequence

$$f_* \otimes_{\mathbb{Z}G} \mathrm{id}_A : \big(P_* \otimes_{\mathbb{Z}G} A, \varepsilon \otimes_{\mathbb{Z}G} A\big) \to \big(P'_* \otimes_{\mathbb{Z}G} A, \varepsilon' \otimes_{\mathbb{Z}G} A\big)$$

is a $\mathbb{Z}$-chain map. Moreover, if A is a ring, then the complex (8) inherits the structure of an A-chain complex with the A-module structure

$$\begin{aligned} A \times \big(P_i \otimes_{\mathbb{Z}G} A\big) &\to P_i \otimes_{\mathbb{Z}G} A \\ \big(a, (p, b)\big) &\mapsto p \otimes (a \cdot b). \end{aligned}$$

One checks that this indeed leads to an A-chain complex.

Corollary 2.7 (fundamental theorem of group homology)**.** *Let G be a group, let A be a $\mathbb{Z}G$-module, and let (P_*, ε) and (P'_*, ε') be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$. Then for every $n \in \mathbb{N}$ one has canonical isomorphisms*

$$H_n(P_* \otimes_{\mathbb{Z}G} A) \cong_{\mathbb{Z}} H_n(P'_* \otimes_{\mathbb{Z}G} A). \tag{9}$$

If A is a ring, then (9) *is an isomorphism of A-modules.*

Proof. By the uniqueness of projective resolutions from Theorem 2.3, there exists a $\mathbb{Z}G$-homotopy equivalence $f_* : (P_*, \varepsilon) \to (P'_*, \varepsilon')$ extending $\mathrm{id}_{\mathbb{Z}}$. Then $f_* \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathrm{A}} : P_* \otimes_{\mathbb{Z}G} A \to P'_* \otimes_{\mathbb{Z}G} A$ is a $\mathbb{Z}$-chain homotopy equivalence (if A is a ring, this becomes an A-chain homotopy equivalence). As chain homology is chain homotopy invariant, this induces canonical isomorphisms for all $n \in \mathbb{N}$

$$H_n(f_* \otimes_{\mathbb{Z}G} \mathrm{id}_A) : H_n(P_* \otimes_{\mathbb{Z}G} A) \cong_{\mathbb{Z}} H_n(P'_* \otimes_{\mathbb{Z}G} A). \qquad \square$$

Definition 2.8 (group homology)**.** Let G be a group, let A be a $\mathbb{Z}G$-module and let (P_*, ε) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (e.g. the simplicial resolution). For every $n \in \mathbb{N}$ we define the *group homology of G with coefficients A in degree n* as the $\mathbb{Z}$-module in equation (9), i.e.,

$$H_n(G; A) := H_n(P_* \otimes_{\mathbb{Z}G} A).$$

This is well-defined by the fundamental theorem of group homology (Corollary 2.7).

One could use the simplicial resolution (or a variant called the *bar resolution*) to compute and study group homology in low degrees and for certain simple classes of groups. For the purpose of this thesis, we prefer to introduce the topological viewpoint of group homology first and then translate the computations into algebraic topology.

2.1.3. Classifying spaces

One particularly important source of projective resolutions comes from topology. Let X be a path-connected CW-complex with fundamental group G and universal covering $\pi : \widetilde{X} \to X$. Then the free deck transformation action of G on $\widetilde{X}$

turns the singular ($\mathbb{Z}$-) chain complex $C_*(\widetilde{X};\mathbb{Z})$ of $\widetilde{X}$ into a $\mathbb{Z}G$-chain complex consisting of free $\mathbb{Z}G$-modules (see Remark 2.2). If $\widetilde{X}$ is additionally contractible, then $C_*(\widetilde{X};\mathbb{Z})$ together with

$$\begin{aligned} C_0(\widetilde{X};\mathbb{Z}) &\to \mathbb{Z} \\ \operatorname{map}(\Delta^0,\widetilde{X}) \ni \sigma &\mapsto 1, \end{aligned}$$

is a free resolution (in particular: projective resolution) of $\mathbb{Z}$ over $\mathbb{Z}G$. If we equip $\widetilde{X}$ with the lifted CW-structure from X, then also the cellular chain complex $C_*^{\mathrm{cell}}(\widetilde{X};\mathbb{Z})$ can be viewed as a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over $\mathbb{Z}G$. A CW-complex with the above properties is called a *classifying space for G*. In this section, we follow Geoghegan's book [Geo07] and Löh's lecture notes [Löh19].

Definition 2.9 (classifying space [Geo07, p. 162])**.** Let G be a group. A *(model of a) classifying space for G* (also denoted BG or $\mathrm{K}(G,1)$) is a path-connected pointed CW-complex (X,x_0) with fundamental group isomorphic to G and with contractible universal covering. We will often leave the isomorphism, the basepoint and the CW-structure implicit, simply saying that "X is a classifying space for G".

Convention 2.10 (basepoints)**.** If (X,x_0) is a classifying space for G and $x_0' \in X$, then (X,x_0') is also a classifying space for G. Hence, unless basepoints are explicitly relevant, we will omit them from the notation. The G-action on the universal cover $\widetilde{X}$ depends on the choice of basepoint and on the isomorphism $\pi_1(X,x_0) \cong G$; following Geoghegan's approach, we tacitly fix such choices once and for all.

By the above construction, we obtain the fundamental result that (untwisted) group homology of a group G coincides with the singular homology of a classifying space for G.

Theorem 2.11 (group homology via classifying spaces [Bro82, Proposition I.4.1])**.** *Let G be a group, let A be a $\mathbb{Z}$-module (with trivial G-action), and let $n \in \mathbb{N}$. Let X be a classifying space for G. Then there exist canonical isomorphisms*

$$H_n(G;A) \cong_{\mathbb{Z}} H_n(X;A) \cong_{\mathbb{Z}} H_n^{\mathrm{CW}}(X;A).$$

Notice that the above is well-defined, as classifying spaces are unique up to homotopy equivalence (Theorem 2.20) and singular/cellular homology is homotopy-invariant ([Hat02, Corollary 2.11.]).

Example 2.12. Let G be a group with classifying space X. Then the homology group in degree 0 is

$$H_0(G;\mathbb{Z}) = H_0(X;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z},$$

since X is path-connected and singular homology in degree 0 is free of rank equal to the number of path-connected components.

Example 2.13 (group homology of the trivial group)**.** "The" one-point space $\{*\}$ is a classifying space for "the" trivial group 1: the fundamental group of $\{*\}$ is trivial and the universal covering $\{*\}$ is contractible. This gives homology groups for $n \in \mathbb{N}_{\geq 1}$

$$H_n(1;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(\{*\};\mathbb{Z}) \cong_{\mathbb{Z}} 0.$$

Example 2.14 (group homology of free Abelian groups)**.** Let $m \in \mathbb{N}_{\geq 1}$. Then the m-torus $\mathbb{T}^m$ is a classifying space for $\mathbb{Z}^m$: the fundamental group of $\mathbb{T}^m$ is isomorphic to $\mathbb{Z}^m$, and the universal covering $\mathbb{R}^m$ is contractible. This gives homology groups for $n \in \mathbb{N}_{\geq 1}$

$$H_n(\mathbb{Z}^m;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(\mathbb{T}^m;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z}^m & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Example 2.15 (group homology of free groups)**.** Let $m \in \mathbb{N}_{\geq 1}$. Then the m-fold wedge $X = \bigvee_{i=1}^m (\mathbb{S}^1, 1)$ of circles is a classifying space for "the" free group F_m of rank m: we have that $\pi_1(X) \cong F_m$ by the Seifert and van Kampen theorem, and the universal covering given by an $2m$-regular tree is contractible. This gives homology groups for $n \in \mathbb{N}_{\geq 1}$

$$H_n(F_m;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(X;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z}^m & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Example 2.16 (group homology of finite cyclic groups)**.** Let $m \in \mathbb{N}_{\geq 2}$. Then the *infinite-dimensional lens space* $X = \mathbb{S}^\infty/(\mathbb{Z}/m)$ is a classifying space for $\mathbb{Z}/m$. Here, $\mathbb{Z}/m$ acts (properly discountinuously) on $\mathbb{S}^\infty$ (regarded as the unit sphere in $\mathbb{C}^\infty$) by scalar multiplication by m-th roots of unity [Hat02, Example 1B.4]. Then, $\mathbb{S}^\infty$ is the contractible universal covering space of X. One computes the homology groups for $n \in \mathbb{N}_{\geq 1}$

$$H_n(\mathbb{Z}/m;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(X;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z}/m & \text{if } n \text{ is odd}, \\ 0 & \text{if } n > 0 \text{ is even}. \end{cases}$$

Note that it is arguably easier to find a concrete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ with $G = \mathbb{Z}/m$ and compute the group homology algebraically [Löh19, Chapter 1.6.3].

Example 2.17 (group homology of surface groups)**.** If S is a closed surface other than $\mathbb{S}^2$ and $\mathbb{R}P^2$, then its universal covering is homotopy equivalent to $\mathbb{R}^2$ and thus contractible [Hat02, Example 1B.2]. In particular, S is a classifying space for its fundamental group.

An *orientable- (resp. non-orientable-) surface group* is a fundamental group of an orientable (resp. non-orientable) closed surface other than $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$). If the surface Σ_g is orientable of genus g, then it has Euler characteristic $\chi(\Sigma_g) = 2 - 2g$ and the associated surface group admits a presentation

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

The group homology of an orientable-surface group is given for $n \in \mathbb{N}_{\geq 1}$ by [Hat02, Example 2.36]

$$H_n\big(\pi_1(\Sigma_g); \mathbb{Z}\big) \cong_{\mathbb{Z}} H_n(\Sigma_g; \mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z}^{2g} & \text{if } n = 1, \\ \mathbb{Z} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

If the surface S_g is non-orientable of genus g, then it has Euler characteristic $\chi(S_g) = 2 - g$ and the associated surface group admits a presentation

$$\pi_1(S_g) \cong \langle a_1, \dots, a_g \mid a_1^2 \cdots a_g^2 \rangle.$$

This can be seen by applying the Seifert and van Kampen theorem to the polygon with pairwise identified edges [Hat02, p.51]. The group homology of a non-orientable-surface group is given for $n \in \mathbb{N}_{\geq 1}$ by [Hat02, Example 2.37]

$$H_n\big(\pi_1(S_g); \mathbb{Z}\big) \cong_{\mathbb{Z}} H_n\big(S_g; \mathbb{Z}\big) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

If $X = \Sigma_0 = \mathbb{S}^2$, then the associated surface group is $\pi_1(X) \cong 1$ and group homology can be computed as in Example 2.13.

If $X = S_1 = \mathbb{R}P^2$, then the associated surface group is $\pi_1(X) \cong \mathbb{Z}/2$ and group homology can be computed as in Example 2.16.

Example 2.18 (Salvetti complex)**.** For a *right-angled Artin group* A_Γ associated to a finite simplicial graph Γ, a classifying space is given by the *Salvetti complex* Sal_Γ, (Definition 3.47). This can be used to compute group homology of right-angled Artin groups (Example 3.48).

Theorem 2.19 (new classifying spaces out of old)**.** *Let G and H be groups and let X and Y be classifying spaces for G and H, respectively. Then:*

1. *If $K \subseteq G$ is a subgroup, then "the" path-connected covering $q : Z \to X$ of X corresponding to the subgroup K of G and equipped with the natural lifted CW-structure, yields a classifying space Z for K.*

2. *The product complex $X \times Y$ is a classifying space for $G \times H$.*

3. *The wedge product $X \vee Y$ is a classifying space for $G * H$.*

Proof. The proof follows from corresponding results in algebraic topology and can be found in the books by Geoghegan [Geo07, Corollary 7.1.4] and Hatcher [Hat02, 1B.5 and 1B.11]. □

Theorem 2.20 (existence of classifying spaces)**.** *Let G be a group. Then G admits a classifying space, which is unique up to pointed homotopy equivalence.*

Sketch of proof. One hands-on way of showing existence comes from attaching cells to the *presentation complex* to kill higher dimensional homology groups (which leads to a contractible universal covering [Geo07, Proposition 7.1.3]). Let $\langle S \mid R \rangle$ be a presentation of a group G. Construct a CW-complex X as

- Take one 0-cell x_0.
- Attach $|S|$-many 1-cells with endpoints x_0.
- Attach $|R|$-many 2-cells to the 1-skeleton according to the relations in R.
- Suppose we have already constructed the n-skeleton X_n for $n \geq 2$. Let A_n be a generating set of $\pi_n(X_n, x_0)$ and choose maps $\gamma_a : \mathbb{S}^n \to X_n$ representing $a \in A_n$, and attach $(n+1)$-cells via γ_a to X_n to obtain X_{n+1}.

One shows using the Seifert and van Kampen theorem that $\pi_1(X, x_0) \cong G$ as well as the Blakers–Massey theorem to show that the universal covering is contractible. Details can be found in Geoghegan's book [Geo07, Proposition 7.1.5, Corollary 7.1.7]. □

Example 2.21 (torsion-free one relator groups)**.** Let G be a *one-relator group*, i.e., G admits a presentation $\langle S \mid r \rangle$ with only one relation. If G is torsion-free (which can be shown to be equivalent to the requirement that r is not a proper power), then the presentation complex achieved in the second step of the proof sketch of Theorem 2.20, is already a classifying space of G [Coc54].

Remark 2.22 (standard simplicial model)**.** Another model of a classifying space of a given group G is the *standard simplicial model*, the "topological origin of the bar resolution" [Löh19, Chapter 4.1.1]. The construction in the proof of Theorem 2.20 has the advantage that it makes transparent that the 2-skeleton can be chosen to coincide with the *presentation complex*. This gives a natural way to generalize finite presentability of groups to higher dimensions (Chapter 2.1.5).

2.1.4. Restriction functor

We now investigate group homology in the context of subgroups, laying important groundwork for our upcoming study of homology growth. For this, we introduce the *restriction functor*, which turns $\mathbb{Z}G$-resolutions into $\mathbb{Z}H$-resolutions for a subgroup $H \subseteq G$ by forgetting the action of elements outside of H.

Definition 2.23 (restriction)**.** Let G be a group and let $i : H \hookrightarrow G$ be a subgroup. Let X be a $\mathbb{Z}G$-module. Then we write $\operatorname{Res}_H^G(X)$ for the $\mathbb{Z}H$-module with underlying Abelian group X and H-action:

$$\begin{aligned} H \times X &\to X \\ (h, x) &\mapsto i(h) \cdot x =: h \cdot x. \end{aligned}$$

Given a $\mathbb{Z}G$-linear map $\Phi : X \to Y$ we define the $\mathbb{Z}H$-linear map

$$\begin{aligned} \operatorname{Res}_H^G(\Phi) : \operatorname{Res}_H^G(X) &\to \operatorname{Res}_H^G(Y) \\ x &\mapsto \Phi(x). \end{aligned}$$

This is functorial: given two $\mathbb{Z}G$-linear maps $\Phi : X \to Y$ and $\Psi : Y \to Z$, one has that

$$\operatorname{Res}_H^G(\Psi \circ \Phi) = \operatorname{Res}_H^G(\Psi) \circ \operatorname{Res}_H^G(\Phi).$$

Example 2.24 (restriction)**.** Let G be a group and let $H \subseteq G$ be a subgroup.

- For the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, the restricted module $\operatorname{Res}_H^G(\mathbb{Z})$ is isomorphic to the trivial $\mathbb{Z}H$-module $\mathbb{Z}$.

- Let $H \subseteq G$ be a subgroup of finite index $d := [G : H]$. Consider a right transversal of H in G, i.e., a subset $\{g_1, \dots, g_d\}$ of G such that

  $$G = \bigsqcup_{l \in \{1,\dots,d\}} H \cdot g_l,$$

  so that $H \setminus G \cong \{g_1, \dots, g_d\}$ as sets. Then we have a canonical isomorphism of $\mathbb{Z}H$-modules given by the unique $\mathbb{Z}H$-linear extension of

  $$\Theta : \bigoplus_{H \setminus G} \mathbb{Z}H \to \operatorname{Res}_H^G(\mathbb{Z}G),\ g_l \mapsto g_l. \tag{10}$$

  Surjectivity. Let $g \in \mathbb{Z}G$ be a $\mathbb{Z}$-basis element. Then g lies in a unique right coset of H, i.e., there exist $h \in H$ and $l \in \{1, \dots, d\}$ such that $g = h \cdot g_l$. Then $\Theta(h \cdot g_l) = h \cdot \Theta(g_l) = h \cdot g_l = g$. In particular, by $\mathbb{Z}$-linearity of Θ, every element $x = \sum_{g \in G} a_g \cdot g$ has a preimage.

Injectivity. Suppose that x is given by the $\mathbb{Z}H$-linear combination

$$x = \sum_{l=1}^{d} \left(\sum_{h \in H} a_h^l \cdot h \right) \cdot g_l \in \bigoplus_{H \backslash G} \mathbb{Z}H,$$

such that $\Theta(x) = 0$ in $\mathbb{Z}G$. Then

$$0 = \Theta(x) = \sum_{l=1}^{d} \left(\sum_{h \in H} a_h^l \cdot h \right) \cdot g_l = \sum_{l=1}^{d} \sum_{h \in H} a_h^l \cdot (h \cdot g_l) \in \mathbb{Z}G.$$

Since the right cosets $Hg_1, \dots, Hg_d$ are pairwise disjoint, the set

$$\{h \cdot g_l \mid l \in \{1, \dots, d\}, h \in H\} \cong G$$

consists of pairwise distinct elements of G. Since $g \in G$ is a $\mathbb{Z}$-basis of $\mathbb{Z}G$, all coefficients a_h^l must vanish. In particular, $x = 0$.

Let us denote by Ψ the inverse isomorphism of Θ. Then given an element $x = \sum_{g \in G} a_g \cdot g \in \operatorname{Res}_H^G(\mathbb{Z}G)$ we have

$$\Psi(x) = \sum_{l=1}^{d} \left(\sum_{h \in H} a_{h \cdot g_l} \cdot h \right) \cdot g_l \in \mathbb{Z}H^d.$$

Lemma 2.25 (restriction of resolutions)**.** *Let G be a group and let H be a subgroup.*

1. Res_H^G *preserves exactness, i.e., if (X_*, ∂_*) is an exact sequence of $\mathbb{Z}G$-modules, then $(\operatorname{Res}_H^G X_*, \operatorname{Res}_H^G \partial_*)$ is an exact sequence of $\mathbb{Z}H$-modules.*

2. Res_H^G *is compatible with direct sums, i.e., for a collection of $\mathbb{Z}G$-modules X_i for $i \in \mathcal{I}$ one has*

$$\operatorname{Res}_H^G \Big(\bigoplus_{i \in \mathcal{I}} X_i \Big) \cong_{\mathbb{Z}H} \bigoplus_{i \in \mathcal{I}} \operatorname{Res}_H^G(X_i).$$

3. *Suppose that $[G : H] < \infty$. If X is finitely $\mathbb{Z}G$-generated by n elements, then $\operatorname{Res}_H^G X$ is finitely $\mathbb{Z}H$-generated by $n \cdot [G : H]$ elements.*

4. *If P is a projective (respectively: free) $\mathbb{Z}G$-module, then $\operatorname{Res}_H^G P$ is a projective (respectively: free) $\mathbb{Z}H$-module.*

5. *If (P_*, ε) is a projective (respectively: free) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then $\big(\operatorname{Res}_H^G P_*, \operatorname{Res}_H^G \varepsilon\big)$ is a projective (respectively: free) resolution of $\mathbb{Z}$ over $\mathbb{Z}H$.*

Proof. Ad 1. The restriction functor Res_H^G is the forgetful functor from $\mathbb{Z}G$-modules to $\mathbb{Z}H$-modules. It acts as the identity on the underlying abelian groups and hence preserves kernels and cokernels. In particular, it is exact: if a sequence of $\mathbb{Z}G$-modules is exact, then so is the sequence obtained by restricting along $H \leq G$.

Ad 2. This follows from the fact that Res_H^G is a forgetful functor, and such functors preserve all colimits. In particular, they preserve direct sums, which are coproducts in module categories.

Ad 3. Suppose that $[G : H] = d$ and X is a finitely generated $\mathbb{Z}G$-module. There exists a surjection $\bigoplus_{i=1}^n \mathbb{Z}G \twoheadrightarrow X$ for some $n \in \mathbb{N}_{\geq 1}$. By the second part and the computation in Example 2.24 we get

$$\operatorname{Res}_H^G\Big(\bigoplus_{i=1}^n \mathbb{Z}G\Big) \cong_{\mathbb{Z}H} \bigoplus_{i=1}^n \operatorname{Res}_H^G(\mathbb{Z}G) \cong_{\mathbb{Z}H} \bigoplus_{i=1}^n \bigoplus_{j=1}^d \mathbb{Z}H.$$

By functoriality, the finitely generated free $\mathbb{Z}H$-module $\operatorname{Res}_H^G\big(\bigoplus_{i=1}^n \mathbb{Z}G\big)$ surjects onto $\operatorname{Res}_H^G X$, so the latter is finitely generated.

Ad 4. By the computation in Example 2.24 and the third part, we get the statement for free modules. Now assume that P is projective, so there exists a $\mathbb{Z}G$-module Q such that $P \oplus Q$ is free. In particular,

$$\operatorname{Res}_H^G(P \oplus Q) \cong_{\mathbb{Z}H} \operatorname{Res}_H^G(P) \oplus \operatorname{Res}_H^G(Q)$$

is a free $\mathbb{Z}H$-module, witnessing that $\operatorname{Res}_H^G(P)$ is a projective $\mathbb{Z}H$-module.

Ad 5. This follows from the fact that $\operatorname{Res}_H^G(\mathbb{Z}) \cong_{\mathbb{Z}H} \mathbb{Z}$ (Example 2.24) and the first and fourth part.

□

2.1.5. Finiteness properties

It is natural to measure the homological complexity of a group in two different ways:

- Up to what degree are its homology groups finitely generated?
- What is the largest degree with non-trivial homology group?

This is encoded by the following *finiteness properties*.

Definition 2.26 (homological finiteness properties)**.** Let $n \in \mathbb{N}$. A group G is

- *of type* FP_n, if there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \in \{0, \dots, N\}$.

- *of type* FP_∞, if there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \in \mathbb{N}$.
- *of type* FP, if there exists a *finite* projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$, i.e., there exists $n \in \mathbb{N}$ such that P_i is finitely generated for $i \in \{0, \dots, N\}$ and $P_i \cong 0$ for all $i \geq n+1$.

We define the *cohomological dimension of* G as

$$\operatorname{cd}(G) := \inf \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{c} \exists \text{ projective resolution } (P_*, \varepsilon) \\ \text{of } \mathbb{Z} \text{ over } \mathbb{Z}G \text{ such that} \\ P_i \cong_{\mathbb{Z}G} 0 \text{ for all } i \geq n+1. \end{array} \right\} \in \mathbb{N} \cup \{\infty\}.$$

Eventhough our version of the definition does not refer to group (co)homology, we use the term *cohomological* dimension to align with the nomenclature in the literature [Bro82, Chapter VIII.2]. We could likewise speak of the *homological* dimension instead, as the following lemma shows.

Lemma 2.27 (relation with group homology)**.** *Let $n \in \mathbb{N}$ and let G be a group.*

1. *If G is of type* FP_{n+1}*, then $H_i(G; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module for all $i \in \{0, \dots, N\}$.*
2. *If $\operatorname{cd}(G) = n$, then $H_n(G; \mathbb{Z})$ is a free $\mathbb{Z}$-module and $H_i(G; \mathbb{Z}) \cong 0$ for all $i \geq n+1$.*
3. *In particular: if $H_n(G; \mathbb{Z}) \not\cong 0$, then $\operatorname{cd}(G) \geq n$.*

Proof. Ad 1. There exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \in \{0, \dots, n+1\}$. Consequently, $P_i \otimes_{\mathbb{Z}G} \mathbb{Z}$ is also finitely generated for all $i \in \{0, \dots, n+1\}$, and in particular

$$H_i(G; \mathbb{Z}) = H_i(P_* \otimes_{\mathbb{Z}G} \mathbb{Z}) = \frac{\ker \partial_i \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}}{\operatorname{Im} \partial_{i+1} \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}}$$

is a quotient of finitely generated $\mathbb{Z}$-modules, thus is itself finitely generated.

Ad 2. There exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that $P_i \cong_{\mathbb{Z}G} 0$ for all $i \geq n+1$. Consequently, also $P_i \otimes_{\mathbb{Z}G} \mathbb{Z} \cong_{\mathbb{Z}G} 0$ and thus $H_i(G; \mathbb{Z}) \cong_{\mathbb{Z}} 0$ for all $i \geq n+1$. Thus

$$H_n(G; \mathbb{Z}) = H_n(P_* \otimes_{\mathbb{Z}G} \mathbb{Z}) = \frac{\ker \partial_n \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}}{\operatorname{Im} \partial_{n+1} \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}} \cong_{\mathbb{Z}} \ker \partial_n \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}} \subseteq P_n \otimes_{\mathbb{Z}G} \mathbb{Z}.$$

This makes $H_n(G; \mathbb{Z})$ a $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $P_n \otimes_{\mathbb{Z}G} \mathbb{Z}$, and the claim follows. $\square$

Working with free modules is easier than with projective modules. In the presence of finiteness properties, one can replace projective resolutions with free ones. This “trick” will simplify our computations in the upcoming chapter.

Proposition 2.28 (free resolutions trick)**.** *Let $n \in \mathbb{N}$ and let G be a group. The following are equivalent:*

1. *G is of type FP_n, i.e., there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \in \{0, \dots, N\}$.*
2. *There exists a free resolution (L_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that L_i is finitely generated for all $i \in \{0, \dots, N\}$.*

Proof. Ad 2. $\Longrightarrow$ 1. This follows from the fact that free modules are projective. *Ad 1. $\Longrightarrow$ 2.* This is a consequence of Schanuel's lemma and is proved in Brown's book [Bro82, Proposition VIII.4.3]. $\square$

A useful way to exhibit projective resolutions with suitable finiteness properties is via classifying spaces. This leads to the a priori stronger notion of the *topological finiteness properties.*

Definition 2.29 (topological finiteness properties)**.** Let $n \in \mathbb{N}$. A group G is

- *of type* F_n, if it admits a classifying space with finite n-skeleton.
- *of type* F_∞, if it admits a classifying space of *finite type*, i.e., whose skeleta are finite in all dimensions.
- *of type* F, if there exists a *finite* classifying space, i.e., a classifying space with finitely many cells.

We define the *geometric dimension of G* as

$$\mathrm{gd}(G) := \inf \bigl\{ \dim_{\mathrm{CW}} X \mid X \text{ is a classifying space for } G \bigr\} \in \mathbb{N} \cup \{\infty\}.$$

Caveat 2.30 (F_n and F_n)**.** It is easy to mix up the notations of free groups F_n and the finiteness type F_n, both being somewhat standard notations in the literature. We avoid possible confusion in two ways: we format the letter ‘F’ slightly differently and we never consider a free group of rank n, and likewise we never consider a group of topological finiteness type F_m.

Remark 2.31 (finiteness properties)**.** Let $n \in \mathbb{N}$ and let G be a group.

- It follows directly from the definitions that
  - type FP $\Longrightarrow$ type FP_∞ $\Longrightarrow$ type FP_n, and

- type F $\Longrightarrow$ type F_∞ $\Longrightarrow$ type F_n.

- It is easy to show that
  - G is of type FP $\Longrightarrow$ $\operatorname{cd}(G) < \infty$, and
  - G is of type F $\Longrightarrow$ $\operatorname{gd}(G) < \infty$.

- By passing to the cellular chain complex of the free G-CW-complex given by the universal covering of a classifying space for G, one sees that the topological finiteness properties "translate" into homological finiteness properties:
  - type F_n implies type FP_n,
  - type F_∞ implies type FP_∞,
  - type F implies type FP,
  - and $\operatorname{cd}(G) \leq \operatorname{gd}(G)$.

  It is an open question known as the *Eilenberg–Ganea problem* asking whether $\operatorname{cd}(G) = \operatorname{gd}(G)$ holds in general. The only way to disprove this would be to find a group with $\operatorname{cd}(G) = 2$ and $\operatorname{gd}(G) = 3$ [Bro82, Theorem VIII.7.1].

- Using the fact that for every group, there exists a classifying space with 2-skeleton being the presentation complex (as sketched in the proof of Theorem 2.20), one can show that [Geo07, Proposition 7.2.1]
  - Every group is of type F_0.
  - G is of type F_1 if and only if G is finitely generated.
  - G is of type F_2 if and only if G is finitely presented.

- For every $n \in \mathbb{N}$, there exist groups that are
  - of type F_n, but not of type FP_{n+1}; and
  - of type FP, but not of type F_2.

  These groups can be constructed as certain subgroups of right-angled Artin groups called *Bestvina–Brady groups* [BB97] (see Remark 3.51).

- By work of Eilenberg–Ganea and Wall, for all finitely presented groups, type FP_n coincides with type F_n for $n \in \mathbb{N}_{\geq 3}$ [Bro82, Chapter VIII.7].

Example 2.32 (finiteness properties)**.**

- "The" trivial group is of type F and is unique with $\operatorname{cd}(G) = \operatorname{gd}(G) = 0$.

- If G is a non-trivial finite group, then G is of type F_∞, but it is obstructed from being of type F. On one hand, the classifying space given by the lens space is a CW-complex of finite type. However, the computation of group homology of cyclic groups (Example 2.16) and the fact that cd is monotonous under passing to subgroups (Theorem 2.33), implies that every torsion group G has $\mathrm{cd}(G) = \mathrm{gd}(G) = \infty$ [Bro82, Corollary VIII.2.5]. In particular, such G is not of type FP by Lemma 2.27.
- For all $n \in \mathbb{N}_{\geq 1}$ we have that $\mathbb{Z}^n$ is of type F and $\mathrm{cd}(G) = \mathrm{gd}(G) = n$, witnessed by the n-torus $\mathbb{T}^n$ as classifying space and the fact that $H_n(\mathbb{Z}^n; \mathbb{Z}) \neq 0$.
- If G is a free group, then $\mathrm{cd}(G) = \mathrm{gd}(G) = 1$, witnessed by a wedge of circles as a 1-dimensional classifying space. It is of type F if and only if it is finitely generated (otherwise, it is only of type F_0). In fact, by the *Stallings–Swan theorem*, for a group G one has that $\mathrm{cd}(G) = 1$ if and only if G is free [Bro82, Chapter VIII.2.2].
- If G is an (orientable) surface group of genus $g \in \mathbb{N}_{\geq 2}$, then G is of type F and $\mathrm{cd}(G) = \mathrm{gd}(G) = 2$. This is witnessed by the existence of a 2-dimensional classifying space, and the fact that $H_2(G; \mathbb{Z}) \neq 0$ (Example 2.17).
- In Chapter 3.2.1, we will see that right-angled Artin groups are of type F and have as geometric dimension the maximal non-trivial clique number of the underlying graph, witnessed by the *Salvetti complex* (Example 3.48).
- Finitely generated torsion-free one-relator groups are of type F and have geometric dimension at most 2 by Example 2.21.

Theorem 2.33 (Finiteness properties of group-theoretic constructions)**.** *Let $n \in \mathbb{N}$ and let G and H be groups of type F_n (resp. FP_n). Then:*

1. *If $K \subseteq G$ is a finite index subgroup, then K is also of type F_n (resp. FP_n) and we have*
$$\mathrm{gd}(K) \leq \mathrm{gd}(G) \text{ and } \mathrm{cd}(K) \leq \mathrm{cd}(G).$$
2. *The direct product $G \times H$ is of type F_n (resp. FP_n) and we have*
$$\mathrm{gd}(G \times H) \leq \mathrm{gd}(G) + \mathrm{gd}(H) \text{ and } \mathrm{cd}(G \times H) \leq \mathrm{cd}(G) + \mathrm{cd}(H).$$
3. *The free product $G * H$ is of type F_n (resp. FP_n)*
$$\mathrm{gd}(G * H) \leq \max\big\{\mathrm{gd}(G), \mathrm{gd}(H)\big\} \text{ and } \mathrm{cd}(G * H) \leq \max\big\{\mathrm{cd}(G), \mathrm{cd}(H)\big\}.$$
4. *The analogous statements hold if G and H are of type F (resp. FP).*

Sketch of proof. For the topological finiteness properties this follows from Theorem 2.19: let X (resp. Y) be classifying spaces witnessing that G (resp. H) is of type F_n. Then we can exhibit classifying spaces of K, $G \times H$ and $G * H$ with finite n-skeleton. Let (P_*, ε) (resp. (Q_*, τ)) be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ (resp. $\mathbb{Z}H$) witnessing that G (resp. H) is of type FP_n. By Lemma 2.25, $\left(\mathrm{Res}_K^G P_*, \mathrm{Res}_K^G \varepsilon\right)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}K$ exhibiting that K is of type FP_n. Similarly, one can show that $\left(P_* \otimes_{\mathbb{Z}} Q_*, \varepsilon \otimes_{\mathbb{Z}} \tau\right)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times H]$ [Bro82, Chapter V.1]. For the part about free products, we refer to Cohen's paper about projective resolutions of graph products [Coh95, Corollary 1]. $\square$

2.2. Homological invariants

Now that we have defined group homology, we aim to extract numerical invariants from the (finitely generated) homology groups. On one hand, for $\mathbb{Z}$-coefficients, we can examine the cardinality of the (finite) torsion subgroup. On the other hand, given any field $\mathbb{F}$, we can look at the dimension of the homology group with $\mathbb{F}$-coefficients. After recalling the basic structure of finitely generated $\mathbb{Z}$-modules, we invoke the universal coefficient theorem to understand the relationship between the above homological invariants.

2.2.1. Structure of modules

The integral homology groups of a group G are $\mathbb{Z}$-modules. As such, their *torsion elements* form a $\mathbb{Z}$-submodule called the *torsion submodule.* When these homology groups are finitely generated, the the torsion submodule is finite and its structure is completely classified by the fundamental structure theorem for finitely generated abelian groups.

Definition 2.34 (torsion element, torsion submodule)**.** Let A be a $\mathbb{Z}$-module.

- We say that an element $x \in A$ is a *torsion element* if there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $m \cdot x = 0$.
- Then $T(A) := \{x \in A \mid x \text{ is torsion}\}$ forms a submodule called the *torsion submodule of A.*
- If $T(A) = \{0\}$, we say that A is *torsion-free.*
- If $A = T(A)$, we say that A is *torsion.*
- We write $\operatorname{tors} A := |T(A)| \in \mathbb{N} \cup \{\infty\}$.

Before considering some examples, we record that torsion is well-behaved under direct sums and passage to submodules.

Proposition 2.35 (torsion under module operations). *Let $n \in \mathbb{N}_{\geq 1}$ and let $A, A_1, \dots, A_n$ and B be $\mathbb{Z}$-modules. Then*

1. *If $i : A \hookrightarrow B$ is an injective $\mathbb{Z}$-module homomorphism, then* $\operatorname{tors} A \leq \operatorname{tors} B$.
2. *We have $T(A_1 \oplus \dots \oplus A_n) \cong_{\mathbb{Z}} T(A_1) \oplus \dots \oplus T(A_n)$. In particular:*

$$\operatorname{tors}\bigl(A_1 \oplus \dots \oplus A_n\bigr) = \operatorname{tors} A_1 \cdot \ldots \cdot \operatorname{tors} A_n.$$

Proof. Ad 1. We show that i restricts to $i_{|_{T(A)}} : T(A) \hookrightarrow T(B)$. Assume that $x \in T(A)$. Then there exists an $m \in \mathbb{Z} \setminus \{0\}$ such that $mx = 0$. Then $m \cdot i(a) = i(ma) = i(0) = 0$ and thus $i(a) \in T(B)$.

Ad 2. We show that $T(A_1 \oplus \dots \oplus A_n) = T(A_1) \oplus \dots \oplus T(A_n)$. First assume that $x = (x_1, \dots, x_n) \in A_1 \oplus \dots \oplus A_n$ is torsion, so there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $mx = 0$. In particular, for all $i \in \{1, \dots, n\}$ one has that $mx_i = 0$, so $x \in T(A_1) \oplus \dots \oplus T(A_n)$. Conversely, assume that $x = (x_1, \dots, x_n) \in T(A_1) \oplus \dots \oplus T(A_n)$. So for every $i \in \{1, \dots, n\}$ there exists an $m_i \in \mathbb{Z} \setminus \{0\}$ such that $m_i x_i = 0$. Consider $m := m_1 \cdot \dots \cdot m_n$. Then $mx = 0$, and in particular $x \in T(A_1 \oplus \dots \oplus A_n)$. □

Example 2.36 (torsion submodules).

- The $\mathbb{Z}$-module $\mathbb{Z}$ is torsion-free: assume that $x \in T(\mathbb{Z})$, so there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $mx = 0$. Since $\mathbb{Z}$ is an integral domain, it follows that $x = 0$. In particular, $\operatorname{tors} \mathbb{Z} = 1$ and by Proposition 2.35 also $\operatorname{tors} \mathbb{Z}^n = 1$.

- The $\mathbb{Z}$-module $\mathbb{Q}$ is torsion-free. Assume that $x = pq^{-1} \in T(\mathbb{Q})$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$: Then there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $nx = \frac{np}{q} = 0$. In particular, $np = 0$ in $\mathbb{Z}$ which implies that $p = 0$. Thus $x = 0$ and therefore $\operatorname{tors} \mathbb{Q} = 1$.

- Let $n \in \mathbb{N}_{\geq 1}$. The $\mathbb{Z}$-module $\mathbb{Z}/n$ is torsion: every element is killed by $n \in \mathbb{Z} \setminus \{0\}$. In particular, $\operatorname{tors} \mathbb{Z}/n = n$.

- The $\mathbb{Z}$-module $\prod_{\mathbb{N}} \mathbb{Z}/2$ is torsion: every element $x \in A$ is killed by $2 \in \mathbb{Z} \setminus \{0\}$. In particular, $\operatorname{tors} A = \infty$.

- Let $\mathbb{F}$ be a field. The $\mathbb{Z}$-module $\mathbb{F}$ is torsion-free if and only if $\operatorname{char} \mathbb{F} = 0$. Assume that $\operatorname{char} \mathbb{F} = p \neq 0$ for p prime. Then $1 \in T(\mathbb{F}) \setminus \{0\}$, witnessed by the scalar $p \cdot 1 = 0$. Conversely, assume that $x \in T(\mathbb{F}) \setminus \{0\}$, so there exists an $m \in \mathbb{Z} \setminus \{0\}$ such that $m \cdot x = 0$. Because $\mathbb{F}$ is a field, x is invertible and $0 = (m \cdot x) \cdot x^{-1} = m \cdot (x \cdot x^{-1}) = m \cdot 1$. In particular, $\operatorname{char} \mathbb{F} \neq 0$.

Theorem 2.37 (finitely generated $\mathbb{Z}$-modules [DF04, Chapter 5.2, Theorem 3])**.** *Let A be a finitely generated $\mathbb{Z}$-module. Then there exist $d, k \in \mathbb{N}$, prime numbers $p_1, \dots, p_k \in \mathbb{N}_{\geq 2}$ and positive numbers $n_1, \dots, n_k \in \mathbb{N}_{\geq 1}$ such that*

$$A \cong_{\mathbb{Z}} \mathbb{Z}^d \oplus \bigoplus_{i \in \{1,\dots,k\}} \mathbb{Z}/p_i^{n_i}. \tag{11}$$

All $d, k, p_1, \dots, p_k$ and $n_1, \dots, n_k$ are uniquely determined by A, up to reordering. In particular, we have

$$T(A) \cong_{\mathbb{Z}} \bigoplus_{i \in \{1,\dots,k\}} \mathbb{Z}/p_i^{n_i} \quad \textit{and} \quad \operatorname{tors} A = \prod_{i \in \{1,\dots,k\}} p_i^{n_i} < \infty.$$

Convention 2.38. Given a finitely generated $\mathbb{Z}$-module A, we will write

$$A \cong_{\mathbb{Z}} \mathbb{Z}^d \oplus \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i},$$

for the decomposition in equation (11), leaving the data of $d, k, p_1, \dots, p_k, n_1, \dots, n_k$ and $\mathcal{I}$ implicit. We call the unique number d the *rank* $\operatorname{rk}_{\mathbb{Z}}(A)$ *of* A. For a prime p and a finitely generated $\mathbb{Z}$-module A with decomposition as above, we write

$$\operatorname{mult}_p A := |\{i \in \mathcal{I} \mid p_i = p\}|\,.$$

Caveat 2.39 (necessity of finite generation)**.** Notice that finite generation of A is necessary for a direct sum decomoposition as in Theorem 2.37: consider the $\mathbb{Z}$-module $\mathbb{Q}$. Assume that it admits a decomposition into a direct sum of a free module and the torsion subgroup. Since $T(\mathbb{Q}) \cong_{\mathbb{Z}} 0$, it follows that $\mathbb{Q} \cong_{\mathbb{Z}} \mathbb{Z}^I$, which is a contradiction because $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

We now turn our attention towards the free part. If we take field coefficients $\mathbb{F}$, then $H_n(G; \mathbb{F})$ inherits the structure of an $\mathbb{F}$-vector space, with a well-defined notion of dimension. If we take $\mathbb{F} = \mathbb{Q}$, this gives rise to the classical *Betti number*. Throughout this section, let p be a prime number.

Lemma 2.40 (properties of dimension)**.** *Let A and B be two $\mathbb{F}$-vector spaces. Let $C \subseteq A$ be a subvector space of A. Then*

1. $\dim_{\mathbb{F}}(A \oplus B) = \dim_{\mathbb{F}} A + \dim_{\mathbb{F}} B$*,*
2. $\dim_{\mathbb{F}}(A \otimes B) = \dim_{\mathbb{F}} A \cdot \dim_{\mathbb{F}} B$*,*
3. $\dim_{\mathbb{F}} C \leq \dim_{\mathbb{F}} A$*, and*
4. $\dim_{\mathbb{F}}(A/C) = \dim_{\mathbb{F}} A - \dim_{\mathbb{F}} C$*, if A is finite-dimensional.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{F}$-bases of A and B, respectively. Then one checks easily that $\mathcal{A} \oplus \mathcal{B} := \{(a, 0) \mid a \in \mathcal{A}\} \cup \{(0, b) \mid b \in \mathcal{B}\}$ and $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B} := \{a \otimes b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ are bases for $A \oplus B$ and $A \otimes B$, respectively. A basis $\mathcal{C}$ of C can be extended to a basis of A by the basis extension theorem; in particular its cardinality is bounded by the cardinality of a basis of A. For the fourth part, apply the rank-nullity theorem to the projection homomorphism [DF04, Chapter 11.1, Theorem 7]. □

Definition 2.41 (ordinary Betti number, mod p Betti number)**.** Let $n \in \mathbb{N}$, let G be a group and let $\mathbb{F}$ be a field. The *n-th $\mathbb{F}$-Betti number of G* is defined as

$$b_n(G; \mathbb{F}) := \dim_{\mathbb{F}} H_n(G; \mathbb{F}).$$

If $\mathbb{F} = \mathbb{Q}$, we call $b_n(G; \mathbb{Q})$ the *(ordinary) n-th ($\mathbb{Q}$-)Betti number of G*. If $\mathbb{F} = \mathbb{F}_p$, we call $b_n(G; \mathbb{F}_p)$ the *n-th mod p (or $\mathbb{F}_p$-)Betti number of G*.

Definition 2.42 (Euler characteristic)**.** Given a group G of type F, its *Euler characteristic* $\chi(G)$ is defined as the finite alternating sum

$$\sum_{n \in \mathbb{N}} (-1)^n \cdot b_n(G; \mathbb{Q}).$$

This notion of Euler characteristic of a group with finite classifying space X coincides with the usual notion of Euler characteristic of finite CW-complexes. As mentioned in the introduction, χ is multiplicative with respect to finite index subgroups.

Theorem 2.43 (proportionality principle [Bro82, Theorem IX.6.4])**.** *Let G be a group of type* F*, and let $H \subseteq G$ be a finite index subgroup. Then*

$$\chi(H) = [G : H] \cdot \chi(G).$$

Remark 2.44 (classifying space and deficiency)**.** Recall the deficiency δ from Chapter 1.2.4. Kar–Nikolov proved the following: if a finitely presented group G has $\mathrm{gd}(G) \leq 2$, then $\delta(G) = 1 - \chi(G)$, and as a consequence of the proportionality principle (Theorem 2.43), $\delta(H) - 1 = [G : H] \cdot (\delta(G) - 1)$ for every finite index subgroup H of G [KN16, Lemma 2]. Thus, $\mathrm{DG}(G) = -\chi(G)$. For example, this shows that for products of free groups with $m, m' \in \mathbb{N}_{\geq 1}$

$$\begin{aligned} \mathrm{DG}(F_m \times F_{m'}) &= -\chi(F_m \times F_{m'}) \\ &= -b_2(F_m \times F_{m'}) + b_1(F_m \times F_{m'}) - b_0(F_m \times F_{m'}) \\ &= -m \cdot m' + (m + m') - 1 \\ &= -(m-1)(m'-1). \end{aligned}$$

$F_m \times F_{m'}$ is a right-angled Artin group, and we postpone the computation of its Betti numbers to Example 3.50.

2.2.2. Universal coefficient theorem

To understand the interplay between torsion, $\mathbb{Q}$-, and $\mathbb{F}_p$-Betti numbers, we need to recall an important theorem. In particular, this will justify why for the study of Betti numbers, it suffices to consider the coefficient fields $\mathbb{Q}$ and $\mathbb{F}_p$.

Definition 2.45 (Tor-functor)**.** Let R be a ring, and let A and B be R-modules. Consider a projective resolution P_* of A over R. Then we define the R-module

$$\operatorname{Tor}_1^R(A, B) := H_1(P_* \otimes_R B).$$

This is well-defined by the fundamental theorem of homological algebra 2.3. If $\mathbb{F}$ is a field (viewed as a $\mathbb{Z}$-module), then $\operatorname{Tor}_1^R(A, B)$ carries the structure of an $\mathbb{F}$-vector space induced by the $\mathbb{F}$-vector space structure on $P_i \otimes_{\mathbb{Z}} \mathbb{F}$ (see Example 2.6).

The following is a basic exercise in homological algebra.

Proposition 2.46 (properties of Tor_1^R [Hat02, Proposition 3A.5])**.** *Let $r, s \in \mathbb{N}_{\geq 1}$, and let $\mathcal{I}$ be an index set. Consider R-modules A, B, and A_i for $i \in \mathcal{I}$. Then the following hold:*

1. $\operatorname{Tor}_1^R(A, B) \cong_R \operatorname{Tor}_1^R(B, A)$.
2. $\operatorname{Tor}_1^R\big(\bigoplus_{i \in \mathcal{I}} A_i, B\big) \cong_R \bigoplus_{i \in \mathcal{I}} \operatorname{Tor}_1^R(A_i, B)$.
3. *If A is a torsion-free R-module, then* $\operatorname{Tor}_1^R(A, B) \cong_R 0$.
4. *If $R = \mathbb{Z}$ and $\mathbb{F}$ is a field with* $\operatorname{char} \mathbb{F} = 0$, *then* $\operatorname{Tor}_1^{\mathbb{Z}}(A, \mathbb{F}) \cong_{\mathbb{F}} 0$.
5. $\operatorname{Tor}_1^R(A, B) \cong_R \operatorname{Tor}_1^R\big(T(A), B\big)$.
6. *If $R = \mathbb{Z}$, then* $\operatorname{Tor}_1^{\mathbb{Z}}\big(\mathbb{Z}/r, A\big) \cong_{\mathbb{Z}} \ker\,(A \xrightarrow{\cdot r} A)$. *In particular:*

$$\operatorname{Tor}_1^{\mathbb{Z}}\big(\mathbb{Z}/r, \mathbb{Z}/s\big) \cong_{\mathbb{Z}} \mathbb{Z}/\gcd(r, s).$$

Theorem 2.47 (universal coefficient theorem [Rot09, Corollary 7.57])**.** *Let $n \in \mathbb{N}$, let G be a group, and let A be a $\mathbb{Z}$-module (with trivial G-action). Then*

$$H_n(G; A) \cong_{\mathbb{Z}} \big(H_n(G; \mathbb{Z}) \otimes_{\mathbb{Z}} A\big) \oplus \operatorname{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(G; \mathbb{Z}), A\big). \tag{12}$$

If $A = \mathbb{F}$ is a field, then the isomorphism (12) *respects the natural $\mathbb{F}$-vector space structures, making it an isomorphism of $\mathbb{F}$-vector spaces.*

Corollary 2.48 (Betti numbers and characteristic)**.** *Let $n \in \mathbb{N}_{\geq 1}$, let $\mathbb{F}$ be a field, and let G be a group of type* FP_n. *Then*

- *If* $\operatorname{char}\mathbb{F} = 0$*, then*
$$b_n(G;\mathbb{F}) = \operatorname{rk}_{\mathbb{Z}} H_n(G;\mathbb{Z}). \tag{13}$$
- *If* $\operatorname{char}\mathbb{F} = p$ *for a prime number* p*, then*
$$b_n(G;\mathbb{F}) = \operatorname{rk}_{\mathbb{Z}} H_n(G;\mathbb{Z}) + \operatorname{mult}_p H_n(G;\mathbb{Z}) + \operatorname{mult}_p H_{n-1}(G;\mathbb{Z}). \tag{14}$$

In particular:

- $b_n(G;\mathbb{F})$ *is determined by the characteristic* $\operatorname{char}\mathbb{F}$*, and*
- *if* $H_n(G;\mathbb{Z})$ *and* $H_{n-1}(G;\mathbb{Z})$ *are* p*-torsion-free for a prime* p*, then*
$$b_n(G;\mathbb{F}) \;=\; b_n(G;\mathbb{Q})$$
for any field $\mathbb{F}$ *with* $\operatorname{char}\mathbb{F} = p$*.*

Proof. Since G is of type FP_n, its homology groups in degrees n and $n-1$ are finitely generated $\mathbb{Z}$-modules by Lemma 2.27. Let us write
$$H_n(G;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}^r \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i} \text{ and } H_{n-1}(G;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}^s \oplus \bigoplus_{j\in\mathcal{J}} \mathbb{Z}/q_j^{m_j}.$$
Assume that $\operatorname{char}\mathbb{F} = 0$. Then $\mathbb{F}$ is a torsion-free $\mathbb{Z}$-module, and by the fourth part of Proposition 2.46, the Tor-term in equation (12) vanishes. Therefore
$$\begin{aligned} H_n(G;\mathbb{F}) &\cong_{\mathbb{F}} H_n(G;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} \\ &\cong_{\mathbb{F}} \Big(\mathbb{Z}^r \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big) \otimes_{\mathbb{Z}} \mathbb{F} \\ &\cong_{\mathbb{F}} (\mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{F}) \oplus \bigoplus_{i\in\mathcal{I}} \big(\mathbb{Z}/p_i^{n_i} \otimes_{\mathbb{Z}} \mathbb{F}\big) \\ &\cong_{\mathbb{F}} \mathbb{F}^r. \end{aligned}$$
In particular, $\dim_{\mathbb{F}} H_n(G;\mathbb{F}) = r = \operatorname{rk}_{\mathbb{Z}} H_n(G;\mathbb{Z})$.

Now assume that $\operatorname{char}\mathbb{F} = p$ for p prime. In the universal coefficient formula (12), we get the contribution
$$\begin{aligned} \operatorname{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(G;\mathbb{Z}), \mathbb{F}\big) &\cong_{\mathbb{F}} \operatorname{Tor}_1^{\mathbb{Z}}\Big(\bigoplus_{j\in\mathcal{J}} \mathbb{Z}/q_j^{m_j}, \mathbb{F}\Big) \\ &\cong_{\mathbb{F}} \bigoplus_{j\in\mathcal{J}} \operatorname{Tor}_1^{\mathbb{Z}}\big(\mathbb{Z}/q_j^{m_j}, \mathbb{F}\big) \\ &\cong_{\mathbb{F}} \bigoplus_{j\in\mathcal{J}} \ker\Big(\mathbb{F} \xrightarrow{\cdot q_j^{m_j}} \mathbb{F}\Big) \\ &\cong_{\mathbb{F}} \mathbb{F}^{|\{j\in\mathcal{J} \mid q_j = p\}|}. \end{aligned}$$

where we have used the sixth part of Proposition 2.46 and that

$$\ker\Bigl(\mathbb{F} \xrightarrow{\cdot q_j^{m_j}} \mathbb{F}\Bigr) \cong_{\mathbb{Z}} \begin{cases} 0 \text{ if } q_j \neq p, \\ \mathbb{F} \text{ if } q_j = p. \end{cases}$$

Similarly, we compute

$$\begin{aligned} H_n(G;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} &\cong_{\mathbb{Z}} \Bigl(\mathbb{Z}^r \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i}\Bigr) \otimes_{\mathbb{Z}} \mathbb{F} \\ &\cong_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F})^r \oplus \bigoplus_{i\in\mathcal{I}} \bigl(\mathbb{Z}/p_i^{n_i} \otimes_{\mathbb{Z}} \mathbb{F}\bigr) \\ &\cong_{\mathbb{Z}} \mathbb{F}^r \oplus \mathbb{F}^{|\{i\in\mathcal{I} | p_i = p\}|}, \end{aligned}$$

where we have used

$$\mathbb{Z}/p_i^{n_i} \otimes_{\mathbb{Z}} \mathbb{F} \cong_{\mathbb{F}} \mathbb{F}/p_i^{n_i} \cong_{\mathbb{F}} \begin{cases} 0 \text{ if } p_i \neq p, \\ \mathbb{F} \text{ if } p_i = p. \end{cases}$$

Combining the above computations with equation (12) we obtain

$$\begin{aligned} \dim_{\mathbb{F}} H_n(G;\mathbb{F}) &= \dim_{\mathbb{F}} \bigl(H_n(G;\mathbb{Z}) \otimes_{\mathbb{Z}} A\bigr) + \dim_{\mathbb{F}} \operatorname{Tor}_1^{\mathbb{Z}}\bigl(H_{n-1}(G;\mathbb{Z}), \mathbb{F}\bigr) \\ &= r + |\{j \in \mathcal{J} \mid q_j = p\}| + |\{i \in \mathcal{I} \mid p_i = p\}| \\ &= \operatorname{rk}_{\mathbb{Z}} H_n(G;\mathbb{Z}) + \operatorname{mult}_p H_n(G;\mathbb{Z}) + \operatorname{mult}_p H_{n-1}(G;\mathbb{Z}). \qquad \square \end{aligned}$$

Remark 2.49 (topological version)**.** Theorem 2.47 holds verbatim when the group G is replaced by a topological space X, interpreting homology accordingly.

Example 2.50 (zeroth Betti number)**.** Let G be a group and let $\mathbb{F}$ be a field. Then

$$H_0(G;\mathbb{F}) \cong_{\mathbb{F}} \bigl(H_0(G;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}\bigr) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(0, \mathbb{F}) \cong_{\mathbb{F}} \mathbb{F},$$

and thus $b_0(G;\mathbb{F}) = 1$.

Example 2.51 (Betti numbers)**.** Let $\mathbb{F}$ be a field. The following are consequences of the computations of group homology in Chapter 2.1.3 and Corollary 2.48:

- The $\mathbb{F}$-Betti number of "the" trivial group is $b_n(1;\mathbb{F}) = 0$ for all $n \in \mathbb{N}_{\geq 1}$.
- Let $m \in \mathbb{N}_{\geq 1}$. Then

$$b_n(\mathbb{Z}^m;\mathbb{F}) = \begin{cases} m & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Let $p \mid m$.	$b_n(\mathbb{Z}/m;\mathbb{Q})$	$\operatorname{mult}_p H_n(\mathbb{Z}/m;\mathbb{Z})$	$\operatorname{mult}_p H_{n-1}(\mathbb{Z}/m;\mathbb{Z})$	$b_n(\mathbb{Z}/m;\mathbb{F}_p)$
$n=0$	1	0	0	1
n odd	0	1	0	1
$n>0$ even	0	0	1	1

Figure 1: Computation of $\mathbb{F}_p$-Betti numbers of finite cyclic groups for $p \mid m$.

- Let $m \in \mathbb{N}_{\geq 1}$. Then

$$b_n(F_m;\mathbb{F}) = \begin{cases} m & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Example 2.52 (Betti numbers of finite cyclic groups)**.** Let $m \in \mathbb{N}_{\geq 1}$ and let p be a prime. Then $b_n(\mathbb{Z}/m;\mathbb{Q}) = 0$ for all $n \in \mathbb{N}_{\geq 1}$. Notice that we have

$$\operatorname{mult}_p(\mathbb{Z}/m) = \begin{cases} 0 & \text{if } p \nmid m, \\ 1 & \text{if } p \mid m. \end{cases}$$

Using formula (14), we compute the mod p Betti numbers for $n \in \mathbb{N}_{\geq 1}$ (Figure 1)

$$b_n(\mathbb{Z}/m;\mathbb{F}_p) = \begin{cases} 1 & \text{if } p \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.53 (Betti numbers of surface groups)**.** Let $g \in \mathbb{N}_{\geq 1}$ and let $\pi_1(\Sigma_g)$ be an orientable-surface group of genus g. Then all of its homology groups are torsion-free and thus for every field $\mathbb{F}$ and $n \in \mathbb{N}_{\geq 1}$

$$b_n\bigl(\pi_1(\Sigma_g);\mathbb{F}\bigr) = \begin{cases} 2g & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

Now let $\pi_1(S_g)$ be a non-orientable-surface group of genus g. Its $\mathbb{Q}$-Betti numbers are given by

$$b_n\bigl(\pi_1(S_g);\mathbb{Q}\bigr) = \begin{cases} g-1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

However, its first homology group contains 2-torsion. In fact:

$$\operatorname{mult}_p H_n(\pi_1(S_g); \mathbb{Z}) \neq 0 \iff n = 1 \text{ and } p = 2.$$

This allows us to compute the $\mathbb{F}_2$-Betti numbers for $n \in \mathbb{N}_{\geq 1}$ using formula (14):

$$b_n\big(\pi_1(S_g); \mathbb{F}_2\big) = \begin{cases} g & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

For $p \neq 2$, the $\mathbb{F}_p$-Betti numbers coincide with the $\mathbb{Q}$-Betti numbers.

2.2.3. Finite index subgroups

For subgroups of finite index, the restriction functor retains sufficient control over the ranks of the modules and the *norms* of the boundary operators, allowing us to estimate the "sizes" of homology groups under passing to finite index subgroups. For the free part, this follows from a rather basic estimate. To control the torsion part, we make use of *Gabber's inequality*, which involves operator norms of boundary maps.

Proposition 2.54 (homology dimension estimate)**.** *Let $\mathbb{F}$ be a field. Consider a sequence of $\mathbb{F}$-vector spaces*

$$X_{n+1} \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1},$$

such that X_n and X_{n-1} are finite-dimensional and $\partial_n \circ \partial_{n+1} = 0$. Then

$$\dim_{\mathbb{F}} H_n(X_*) \leq \dim_{\mathbb{F}} X_n.$$

Proof. Using the dimension formula for quotient spaces and the fact that $\ker \partial_n$ is a subspace of X_n we get by Lemma 2.40 that

$$\dim_{\mathbb{F}} H_n(X_*) = \dim_{\mathbb{F}} \left(\frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} \right) = \dim_{\mathbb{F}} \ker \partial_n - \dim_{\mathbb{F}} \operatorname{Im} \partial_{n+1} \leq \dim_{\mathbb{F}} X_n. \qquad \square$$

Theorem 2.55 (dimension of group homology and finite index subgroups)**.** *Let $N \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G be a group of type FP_N. For every $k \in \{0, \ldots, N\}$ there exists a constant $c_k(G) \in \mathbb{R}_{\geq 0}$ such that for every finite index subgroup $H \subseteq G$:*

$$b_k(H; \mathbb{F}) = \dim_{\mathbb{F}} H_k(H; \mathbb{F}) \leq [G : H] \cdot c_k(G).$$

Proof. As G is of type FP_N, by Proposition 2.28 there exists a free resolution (L_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that $L_0, \dots, L_N$ are finitely generated free. By choosing bases we may write $L_0 \cong_{\mathbb{Z}G} \mathbb{Z}G^{r_0}, \dots, L_N \cong_{\mathbb{Z}G} \mathbb{Z}G^{r_N}$ for $r_0, \dots, r_N \in \mathbb{N}_{\geq 1}$. By Lemma 2.25, $\left(\mathrm{Res}_H^G L_*, \mathrm{Res}_H^G \varepsilon\right)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ such that $\mathrm{Res}_H^G L_0 \cong_{\mathbb{Z}H} \mathbb{Z}H^{[G:H]\cdot r_0}, \dots, \mathrm{Res}_H^G L_N \cong_{\mathbb{Z}H} \mathbb{Z}H^{[G:H]\cdot r_N}$ are finitely generated free. Hence, for every $k \in \{0, \dots, n\}$

$$\begin{aligned}
\dim_{\mathbb{F}} H_k(H; \mathbb{F}) &= \dim_{\mathbb{F}} H_k\big((\mathrm{Res}_H^G L_*) \otimes_{\mathbb{Z}H} \mathbb{F}\big) && \text{(fundamental theorem 2.7)} \\
&\leq \dim_{\mathbb{F}} \big((\mathrm{Res}_H^G L_k) \otimes_{\mathbb{Z}H} \mathbb{F}\big) && \text{(Proposition 2.54)} \\
&= \dim_{\mathbb{F}} \big(\mathbb{Z}H^{[G:H]\cdot r_k} \otimes_{\mathbb{Z}H} \mathbb{F}\big) && \text{(Lemma 2.25)} \\
&= \dim_{\mathbb{F}} \mathbb{F}^{[G:H]\cdot r_k} \\
&= [G : H] \cdot r_k.
\end{aligned}$$

Notice that $c_k(G) := r_k$ only depends on G and k. $\square$

To estimate the torsion part, we need a more refined approach, as torsion is ill-behaved under quotient modules. The main tool to bound the torsion in the group homology of a finite index subgroup is *Gabber's estimate.*

Theorem 2.56 (Gabber's estimate [ABFG24, Proposition 9.1])**.** *Let $N \in \mathbb{N}$ and let (X_*, ∂_*) be a $\mathbb{Z}$-chain complex, such that $X_0, \dots, X_{N+1}$ are finitely generated based free $\mathbb{Z}$-modules. Then for all $k \in \{0, \dots, N\}$*

$$\log \operatorname{tors} H_k(X_*) \leq \mathrm{rk}_{\mathbb{Z}}(X_k) \cdot \log_+ \|\partial_{k+1}\| . \tag{15}$$

We write $\log_+ := \max\{\log, 0\}$ *and*

$$\|\partial_{k+1}\| = \max_{i \in \{1, \dots, n\}} \Big\{ \sum_{j=1}^{m} |M_{ij}| \Big\} \in \mathbb{Z}_{\geq 0}, \tag{16}$$

where M denotes the matrix $M = (M_{ij})_{i,j} \in \mathrm{Mat}(m \times n, \mathbb{Z})$ representing ∂_{k+1} with respect to the chosen bases on X_{k+1} and X_k.

Proof. Fix $k \in \{0, \dots, N\}$ and denote $\partial := \partial_{k+1}$. Identify $X_{k+1} \cong_{\mathbb{Z}} \mathbb{Z}^n$ and $X_k \cong_{\mathbb{Z}} \mathbb{Z}^m$ for $n, m \in \mathbb{N}_{\geq 1}$ via the isomorphisms induced by chosen the bases. The boundary map ∂ is of the form

$$\partial : \mathbb{Z}^n \to \mathbb{Z}^m,$$

given by multiplication with a matrix $M \in \mathrm{Mat}(m \times n, \mathbb{Z})$.

Claim. It holds that $\operatorname{tors} \operatorname{coker}(\partial) \leq \max(\|\partial\|, 1)^m$.

Using the claim we deduce

$$H_k(X_*) = \frac{\ker(\partial_k)}{\operatorname{Im}(\partial_{k+1})} \subseteq \frac{X_k}{\operatorname{Im}(\partial_{k+1})} = \operatorname{coker}(\partial_{k+1}).$$

and inequality (15) follows from applying $\log \operatorname{tors}$ and using the fact that tors is monotonous with respect to submodules (Proposition 2.35).

Proof of Claim. We follow the proof given by Kar-Kropholler-Nikolov [KKN17, Lemma 6]. Note that $\|\partial\| < 1$ is equivalent to $\partial \equiv 0$. In this case

$$\operatorname{tors} \operatorname{coker}(0) = \operatorname{tors} \mathbb{Z}^m = 1 \leq 1^m.$$

Now suppose that $\|\partial\| \geq 1$. Denote by $r = \operatorname{rk} M$ the rank of M. Then by the theory of Smith normal form, $\operatorname{tors} \operatorname{coker}(\partial)$ is the greatest common divisor of determinants of all $r \times r$ minors of M [Jac85, Theorem 3.9]. Therefore, it suffices to bound the size of the determinant of just one $r \times r$ minor. Thus we may assume without loss of generality that $n = m = r$ and that $\operatorname{coker} \partial$ is torsion, i.e., that $\det M \neq 0$. The claim follows from the Leibniz formula and triangle inequality

$$|\det M| \leq \sum_{\pi \in \mathrm{Sym}_r} \prod_{i=1}^{r} \left|M_{i,\pi(i)}\right| \leq \prod_{i=1}^{r} \sum_{j=1}^{r} |M_{ij}| \leq \|\partial\|^r . \qquad \square$$

Our definition of $\|\partial_{k+1}\|$ in (16) coincides with the *operator norm of* ∂_{k+1} with respect to the ℓ^1*-norms on* X_{k+1} *and* X_k induced by the absolute value norm $|-|$ on $\mathbb{Z}$. Gabber's original result was more refined and phrased for ℓ^2-norms [Sou99]. The main goal for the remainder of this section is to prove the estimate in Theorem 2.60 by applying Gabber's inequality (15) to the chain complex of coinvariants of a restricted projective resolution. For this, the key step will be the bound:

$$\left\|(\operatorname{Res}^G_H \Phi) \otimes_{\mathbb{Z}H} \operatorname{id}_{\mathbb{Z}}\right\|_1^{\mathbb{Z}} \leq \left\|\operatorname{Res}^G_H \Phi\right\|_1^{\mathbb{Z}H} = \|\Phi\|_1^{\mathbb{Z}G} < \infty,$$

where Φ is an arbitrary $\mathbb{Z}G$-linear map between finitely generated free $\mathbb{Z}G$-modules. The norm $\|-\|_1^{\mathbb{Z}}$ on the left-hand side is the operator norm of $\mathbb{Z}$-linear maps as defined above in (16). We will now define the norms $\|-\|_1^{\mathbb{Z}G}$ that we use for homomorphisms of modules over the group ring. We do this analogously to (18), but we first need to fix an analogue of the absolute value of $|-|_{\mathbb{Z}}$ on $\mathbb{Z}G$. This is in turn given by the ℓ^1-norm on $\mathbb{Z}G$, viewed as a free $\mathbb{Z}$-module. For a broader account of ℓ^1-norms we refer the reader to textbooks on functional analysis.

Definition 2.57 (ℓ^1-norms)**.** Let G be a group. For $x = \sum_{g \in G} a_g \cdot g \in \mathbb{Z}G$, we define the finite value

$$|x|^{\mathbb{Z}G} := \sum_{g \in G} |a_g|^{\mathbb{Z}} \in \mathbb{Z}_{\geq 0}, \tag{17}$$

where $|-|^{\mathbb{Z}}$ denotes the usual absolute value on $\mathbb{Z}$. For $R \in \{\mathbb{Z}, \mathbb{Z}G\}$, let X be a finitely generated based free R-module with basis $(e_1, \dots, e_n)$. For an element $x = \sum_{i=1}^n x_i \cdot e_i \in X$ with $x_1, \dots, x_n \in R$, we define the *ℓ^1-norm of x (with respect to $|-|^R$ and the chosen basis)* as

$$\|x\|_1^R := \sum_{i=1}^n |x_i|^R \in \mathbb{Z}_{\geq 0}.$$

Given a based finitely generated free R-module Y with basis $(f_1, \dots, f_m)$ and a R-linear map $\Phi : X \to Y$, we define the *operator norm of Φ (with respect to the ℓ^1-norms on X and Y)* as

$$\|\Phi\|_1^R := \max\left\{ \|\Phi(e_1)\|_1^R, \dots, \|\Phi(e_n)\|_1^R \right\} \in \mathbb{Z}_{\geq 0}. \tag{18}$$

In particular, if we have $v_1, \dots, v_n \in R^m$ such that $M = (v_1|\dots|v_n) \in \operatorname{Mat}(m \times n; R)$ is the representing matrix of Φ with respect to the chosen bases, then

$$\begin{aligned}
\|\Phi\|_1^R &= \max_{i\in\{1,\dots,n\}} \left\{ \|\Phi(e_i)\|_1^R \right\} \\
&= \max_{i\in\{1,\dots,n\}} \left\{ \left\| \sum_{j=1}^m M_{ij} \cdot f_j \right\|_1^R \right\} \\
&= \max_{i\in\{1,\dots,n\}} \left\{ \sum_{j=1}^m |M_{ij}|_1^R \right\} \\
&= \max_{i\in\{1,\dots,n\}} \left\{ \|v_i\|_1^R \right\}.
\end{aligned}$$

We see that, the definition (16) of $\|\partial\|$ in Gabber's inequality coincides with the operator norm $\|\partial\|_1^{\mathbb{Z}}$ associated to the ℓ^1-norms on the chain $\mathbb{Z}$-modules.

Lemma 2.58 (first estimate)**.** *Let $\Phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$ be a $\mathbb{Z}G$-linear map. Then*

$$\|\Phi \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}\|_1^{\mathbb{Z}} \leq \|\Phi\|_1^{\mathbb{Z}G}.$$

Proof. Let $M \in \operatorname{Mat}(m \times n; \mathbb{Z}G)$ be the representing matrix for Φ with respect to the standard $\mathbb{Z}G$-bases $(e_1, \dots, e_n)$ and $(f_1, \dots, f_m)$. Then the representing matrix for $\Phi \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}$ with respect to the inherited bases is of the form $\big(\varepsilon(M_{ij})\big)_{i,j} \in \operatorname{Mat}(m \times n; \mathbb{Z})$, where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ denotes the canonical $\mathbb{Z}$-linear augmentation map (induced by $G \ni g \mapsto 1$). This can be seen as follows: let $e_i \otimes 1$ be a standard

$\mathbb{Z}$-basis vector of $\mathbb{Z}G^n \otimes_{\mathbb{Z}G} \mathbb{Z}$ for $i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
(\Phi \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}})(e_i \otimes 1) &= \Phi(e_i) \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}(1) && \text{(definition of } - \otimes_{\mathbb{Z}G} -) \\
&= \Big(\sum_{j=1}^{m} M_{ij} \cdot f_j\Big) \otimes_{\mathbb{Z}G} 1 && \text{(choice of } M) \\
&= \sum_{j=1}^{m} M_{ij} \cdot \big(f_j \otimes_{\mathbb{Z}G} 1\big) && \text{(bilinearity)} \\
&= \sum_{j=1}^{m} f_j \otimes_{\mathbb{Z}G} \big(M_{ij} \cdot 1\big) && \text{(bilinearity)} \\
&= \sum_{j=1}^{m} f_j \otimes_{\mathbb{Z}G} \varepsilon(M_{ij}) && (\mathbb{Z} \text{ is trivial } \mathbb{Z}G\text{-module}) \\
&= \sum_{j=1}^{m} \varepsilon(M_{ij}) \cdot f_j. && \text{(identification } \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \cong \mathbb{Z})
\end{aligned}$$

In particular, if we write $M = (v_1| \dots |v_n)$ for $v_1, \dots, v_n \in \mathbb{Z}G^m$, then

$$\begin{aligned}
\|\Phi \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}\|_1^{\mathbb{Z}} &= \max\{\|\varepsilon(v_1)\|_1^{\mathbb{Z}}, \dots, \|\varepsilon(v_n)\|_1^{\mathbb{Z}}\} \\
&\leq \max\{\|v_1\|_1^{\mathbb{Z}G}, \dots, \|v_n\|_1^{\mathbb{Z}G}\} \\
&= \|\Phi\|_1^{\mathbb{Z}G},
\end{aligned}$$

where $\varepsilon(v_i)$ denotes the vector where we apply ε to each entry. Above we have used the fact that for each entry $x = \sum_{g \in G} a_g \cdot g \in \mathbb{Z}G$ of the vectors $v_i \in \mathbb{Z}G^m$ one has that

$$|x|^{\mathbb{Z}G} = \sum_{g \in G} |a_g|^{\mathbb{Z}} \geq \left|\sum_{g \in G} a_g\right|^{\mathbb{Z}} = |\varepsilon(x)|^{\mathbb{Z}}.$$

This immediately implies that for every vector $v = (x_1, \dots, x_m) \in \mathbb{Z}G^m$ one has

$$\|v\|_1^{\mathbb{Z}G} = \sum_{j=1}^{m} |x_j|^{\mathbb{Z}G} \geq \sum_{j=1}^{m} |\varepsilon(x_j)|^{\mathbb{Z}} = \|\varepsilon(v)\|_1^{\mathbb{Z}}. \qquad \square$$

Lemma 2.59 (second estimate)**.** *Let G be a group, and let $\Phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$ be a $\mathbb{Z}G$-linear map with $m, n \in \mathbb{N}_{\geq 1}$. Equip $\mathbb{Z}G^n$ and $\mathbb{Z}G^m$ with the canonical $\mathbb{Z}G$-bases $(e_1, \dots, e_n)$ and $(f_1, \dots, f_m)$, respectively. Let $H \subseteq G$ be a finite index subgroup with $d := [G : H]$ and choose a right transversal $\{g_1, \dots, g_d\}$ of H in G. Identify $\mathrm{Res}_H^G(\mathbb{Z}G^n) \cong_{\mathbb{Z}H} \mathbb{Z}H^{n \cdot d}$ and $\mathrm{Res}_H^G(\mathbb{Z}G^m) \cong_{\mathbb{Z}H} \mathbb{Z}H^{m \cdot d}$ via choosing bases*

$$\{g_1 \cdot e_1, g_2 \cdot e_1, \dots, g_d \cdot e_n\} \text{ and } \{g_1 \cdot f_1, g_2 \cdot f_1, \dots, g_d \cdot f_m\}, \text{ respectively.}$$

Then we have that $\left\|\mathrm{Res}_H^G \Phi\right\|_1^{\mathbb{Z}H} = \|\Phi\|_1^{\mathbb{Z}G}$.

Proof. Step 1. We prove that the $\mathbb{Z}H$-isomorphism (10)

$$\Psi : \mathrm{Res}_H^G(\mathbb{Z}G) \cong_{\mathbb{Z}H} \mathbb{Z}H^d$$

is an ℓ^1*-isometry*, i.e., that for every $x \in \mathbb{Z}G$ one has that

$$\|x\|_1^{\mathbb{Z}G} = \|\Psi(x)\|_1^{\mathbb{Z}H}.$$

Indeed, we compute for $x = \sum_{g\in G} a_g \cdot g \in \mathbb{Z}G$:

$$\begin{aligned}
\|\Psi(x)\|_1^{\mathbb{Z}H} &= \left\|\sum_{l=1}^{d}\left(\sum_{h\in H} a_{h\cdot g_l}\cdot h\right)\cdot g_l\right\|_1^{\mathbb{Z}H} && \text{(definition of }\Psi)\\
&= \sum_{l=1}^{d}\left|\sum_{h\in H} a_{h\cdot g_l}\cdot h\right|^{\mathbb{Z}H} && \text{(definition of }\|-\|_1^{\mathbb{Z}H})\\
&= \sum_{l=1}^{d}\sum_{h\in H}|a_{h\cdot g_l}|^{\mathbb{Z}} && \text{(definition of }|-|^{\mathbb{Z}H})\\
&= \sum_{g\in G}|a_g|^{\mathbb{Z}} && (g_1,\dots,g_d\text{ is a right transversal})\\
&= \left\|\sum_{g\in G} a_g\cdot g\right\|_1^{\mathbb{Z}G} && \text{(definition of }\|-\|_1^{\mathbb{Z}G})\\
&= \|x\|_1^{\mathbb{Z}G}.
\end{aligned}$$

Step 2. We show that for every $i \in \{1,\dots,n\}$ and $l \in \{1,\dots,d\}$

$$\left\|\mathrm{Res}_H^G(\Phi)(g_l\cdot e_i)\right\|_1^{\mathbb{Z}H} = \|\Phi(g_l\cdot e_i)\|_1^{\mathbb{Z}G}.$$

Let

$$\Phi(g_l\cdot e_i) = \sum_{j=1}^{m} y_j\cdot f_j \in \mathbb{Z}G^m.$$

Then using the fact that Ψ is an isometric $\mathbb{Z}H$-isomorphism we get that

$$\begin{aligned}
\left\|\mathrm{Res}_H^G(\Phi)(g_l\cdot e_i)\right\|_1^{\mathbb{Z}H} &= \|\Phi(g_l\cdot e_i)\|_1^{\mathbb{Z}H} && \text{(definition of }\mathrm{Res}_H^G)\\
&= \left\|\sum_{j=1}^{m} y_j\cdot f_j\right\|_1^{\mathbb{Z}H} && \text{(choice of }y_j)\\
&= \left\|\bigoplus_{j=1}^{m}\Psi\left(\sum_{j=1}^{m} y_j\cdot f_j\right)\right\|_1^{\mathbb{Z}H} && (\Psi\text{ is }\mathbb{Z}H\text{-isometry})
\end{aligned}$$

$$\begin{aligned}
&= \left\| \sum_{j=1}^{m} \Psi(y_j) \cdot f_j \right\|_1^{\mathbb{Z}H} && \text{(definition of } \Psi) \\
&= \sum_{j=1}^{m} \|\Psi(y_j)\|_1^{\mathbb{Z}H} && \text{(definition of } \|-\|_1^{\mathbb{Z}H}) \\
&= \sum_{j=1}^{m} \|y_j\|_1^{\mathbb{Z}G} && (\Psi \text{ is } \mathbb{Z}H\text{-isometry})) \\
&= \left\| \sum_{j=1}^{d} y_j \cdot f_j \right\|_1^{\mathbb{Z}G} && \text{(definition of } \|-\|_1^{\mathbb{Z}G}) \\
&= \|\Phi(g_l \cdot e_i)\|_1^{\mathbb{Z}G}. && \text{(choice of } y_j)
\end{aligned}$$

Step 3. We conclude:

$$\begin{aligned}
\left\|\mathrm{Res}_H^G \Phi\right\|_1^{\mathbb{Z}H} &= \max_{\substack{1\le i\le n\\ 1\le l\le d}} \left\|\mathrm{Res}_H^G(\Phi)(g_l \cdot e_i)\right\|_1^{\mathbb{Z}H} && \text{(definition of operator norm)} \\
&= \max_{\substack{1\le i\le n\\ 1\le l\le d}} \|\Phi(g_l \cdot e_i)\|_1^{\mathbb{Z}G} && \text{(Step 2.)} \\
&= \max_{\substack{1\le i\le n\\ 1\le l\le d}} \|g_l \cdot \Phi(e_i)\|_1^{\mathbb{Z}G} && (\Phi \text{ is } \mathbb{Z}G\text{-linear}) \\
&= \max_{1\le i\le n} \|\Phi(e_i)\|_1^{\mathbb{Z}G} && (G \text{ acts by isometries}) \\
&= \|\Phi\|_1^{\mathbb{Z}G}. && \text{(definition of operator norm)}
\end{aligned}$$

□

Theorem 2.60 (torsion of group homology and finite index subgroups)**.** *Let $N\in\mathbb{N}$ and let G be a group of type FP_{N+1}. For every $k \in \{0,\dots,N\}$ there exists a constant $d_k(G) \in \mathbb{R}_{\ge 0}$ such that for every finite index subgroup $H \subseteq G$:*

$$\log \operatorname{tors} H_k(H;\mathbb{Z}) \le c_k(G) \cdot [G:H].$$

Proof. As G is of type FP_{N+1}, by Proposition 2.28 there exists a free resolution (L_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that $L_0, \dots, L_{N+1}$ are finitely generated free. By choosing bases, we may identify $L_0 \cong_{\mathbb{Z}G} \mathbb{Z}G^{r_0}, \dots, L_{N+1} \cong_{\mathbb{Z}G} \mathbb{Z}G^{r_{N+1}}$. By Lemma 2.25, $(\mathrm{Res}_H^G L_*, \mathrm{Res}_H^G \varepsilon)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ such that

$$\mathrm{Res}_H^G L_0 \cong_{\mathbb{Z}H} \mathbb{Z}H^{[G:H]\cdot r_0}, \dots, \mathrm{Res}_H^G L_{N+1} \cong_{\mathbb{Z}H} \mathbb{Z}H^{[G:H]\cdot r_{N+1}}$$

are finitely generated free $\mathbb{Z}H$-modules. Using our two bounds from Lemma 2.58 and Lemma 2.59 we obtain that for every $k \in \{0,\dots,n\}$

$$\left\|(\mathrm{Res}_H^G \partial_{k+1}) \otimes_{\mathbb{Z}H} \mathrm{id}_{\mathbb{Z}}\right\| \le \left\|\mathrm{Res}_H^G \partial_{k+1}\right\| = \|\partial_{k+1}\| < \infty. \tag{19}$$

Finally, using the fundamental theorem of group homology 2.7, Gabber's inequality (15) and the bound on the norm (19) we obtain

$$\begin{aligned}\log \operatorname{tors} H_k(H;\mathbb{Z}) &= \log \operatorname{tors} H_k\big(\operatorname{Res}_H^G L_* \otimes_{\mathbb{Z}H} \mathbb{Z}\big)\\ &\leq \operatorname{rk}_{\mathbb{Z}} \big(\operatorname{Res}_H^G L_k \otimes_{\mathbb{Z}H} \mathbb{Z}\big) \cdot \log_+ \big\|(\operatorname{Res}_H^G \partial_{k+1}) \otimes_{\mathbb{Z}H} \operatorname{id}_{\mathbb{Z}}\big\|\\ &\leq r_k \cdot [G:H] \cdot \log_+ \|\partial_{k+1}\|\,.\end{aligned}$$

Notice that $d_k(G) := r_k \cdot \log_+ \|\partial_{k+1}\|$ is finite and only depends on G and k. □

Assuming G to have the topological finiteness property type F_{N+1} (which is stronger than FP_{N+1}), we can re-interpret our proofs topologically. Given a classifying space X of G with finite $(N+1)$-skeleton, we apply the results to the cellular chain complex of X.

Corollary 2.61 (homology of finite index subgroups, topological version)**.** *Let $N \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G be a group together with a classifying space X with finite $(N+1)$-skeleton. Let $H \subseteq G$ be a finite-index subgroup. Then for every $k \in \{0,\dots,n\}$*

1. $b_k(H;\mathbb{F}) \leq [G:H] \cdot |\{k\text{-cells of } X\}|$;
2. $\log \operatorname{tors} H_k(H;\mathbb{Z}) \leq |\{k\text{-cells of } X\}| \cdot \log_+ \big\|\partial_{k+1}^{\mathrm{cell}}\big\| \cdot [G:H]$.

Proof. Apply (the proofs of) Theorems 2.55 and 2.60 to the cellular chain complex of X. Let Y be the $[G:H]$-sheeted covering of X corresponding to $H \subseteq G$ with the lifted CW-structure. Then

$$c_k(G) = \dim_{\mathbb{F}} C_k^{\mathrm{cell}}(X;\mathbb{F}) = |\{k\text{-cells of } X\}|\,,$$

and similarly

$$d_k(G) = |\{k\text{-cells of } X\}| \cdot \log_+ \big\|\partial_{k+1}^{\mathrm{cell}}\big\|\,.$$ □

Remark 2.62 (improving the bounds by rebuilding)**.** Notice that the constants $c_k(G)$ and $d_k(G)$ depend on the chosen projective (or free) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, which is used to pass to the restricted projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ to compute group homology of H. This choice always works, but the fundamental theorem of group homology allows us to freely choose another resolution. In some cases there are much "smaller" (in terms of ranks and operator norms of boundary maps) projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}H$, which can be used to obtain better bounds (Example 2.63). If this can be done systematically for finite index subgroups of a group G, such that the new ("rebuilt") resolutions have sufficiently controlled ranks and boundary operator norms ("cheaply"), this can be used to imply the vanishing of Betti numbers growth or torsion growth (Remark 3.27).

Example 2.63 (rebuilding coverings of the circle)**.** Consider a finite d-sheeted covering Y of the circle $X = \mathbb{S}^1$ (with CW-structure given by one 0-cell and one 1-cell). Applying Theorem 2.55 to the cellular chain complex gives

$$b_1(Y;\mathbb{F}) \leq \dim_{\mathbb{F}} C_1^{\mathrm{cell}}(Y;\mathbb{F}) = \dim_{\mathbb{F}} C_1^{\mathrm{cell}}(X;\mathbb{F}) \cdot d = d.$$

Instead of using the lifted CW-structure on Y, we can replace it by a homotopy equivalent space Y' containing only one 0-cell and one 1-cell (Y is itself homeomorphic to $\mathbb{S}^1$). In particular, we can improve the bound above to

$$b_1(Y;\mathbb{F}) = b_1(Y';\mathbb{F}) \leq \dim_{\mathbb{F}} C_1^{\mathrm{cell}}(Y';\mathbb{F}) = 1.$$

This is a meaningful improvement, as the bound above was linear in the index d, while the new bound is sublinear (in fact, constant). This rebuilding argument can be translated into algebra [LLMSU24, Examples 2.6, 2.7].

3. Growth of homological invariants

With the notions of invariant growth and homological invariants of groups in place, we examine the growth of homological invariants (or *homology growth*), namely *torsion growth* and *Betti number growth*. In Chapter 3.1 we compute them for easy examples and observe that torsion growth vanishes in all of them. By the universal coefficient theorem, a difference in the growth of Betti numbers with $\mathbb{Q}$ and $\mathbb{F}_p$ coefficients implies positive torsion growth. We observe that n-th homology growth is bounded for groups of type FP_{n+1}, and apply previous axiomatic results on invariant growth of free products of groups. We state *Lück's approximation theorem* and highlight its central role in the study of homology growth. In Chapter 3.2, we briefly introduce *right-angled Artin groups*, which are groups associated to simplicial graphs whose structural properties often reflect the topology of the *underlying flag complex*. We cite the theorems by Avramidi, Okun and Schreve, and use them to construct a group with positive torsion growth. In Chapter 3.3, we state the *Künneth theorem*, which describes group homology of direct products. Examining dimensions and torsion in this formula relates Betti number growth and torsion growth of direct products to the growth of the factors.

Overview of this chapter.

3.1. Betti number growth and torsion growth

We define the growth of homological invariants and compute them in some simple examples. Using the universal coefficient theorem, we relate torsion growth and Betti number growth with coefficients $\mathbb{Q}$ and $\mathbb{F}_p$. The results from Chapter 2.2.3 imply that homology growth in degree n is bounded for groups of type FP_{n+1}. From Theorem 1.53, we deduce that $\widehat{t_n}(-)$ and $\widehat{b_n}(-)$ are (almost) additive under free products, under suitable assumptions. Lück's approximation theorem provides an interpretation of $\mathbb{Q}$-Betti number growth and could be regarded as the starting point of the study of invariant growth. In the words of Abért–Bergeron–Frączyk–Gaboriau, its discovery in 1994 "naturally led to the study of the growth of other homological invariants as well" [ABFG24, p. 2].

3.1.1. Betti number growth

Betti number growth is the growth of Betti numbers of a group.

Definition 3.1 (Betti number growth)**.** Let G be a residually finite group, and let $N_* \in \mathcal{R}(G)$ be a residual chain. Let $n \in \mathbb{N}$ and let p be a prime. Then the *n-th $\mathbb{Q}$-Betti number growth of G with respect to N_** is defined as

$$\widehat{b_n}(G, N_*; \mathbb{Q}) := \limsup_{k\to\infty} \frac{b_n(N_k;\mathbb{Q})}{[G:N_k]},$$

and the *n-th mod p (or $\mathbb{F}_p$-)Betti number growth of G with respect to N_** is defined as

$$\widehat{b_n}(G, N_*; \mathbb{F}_p) := \limsup_{k\to\infty} \frac{b_n(N_k;\mathbb{F}_p)}{[G:N_k]}.$$

We introduce the following notation

$$\mathbf{H}_n(\mathbb{F}) := \left\{ G \text{ res. fin.} \;\middle|\; \widehat{b_j}(G, N_*; \mathbb{F}) = 0 \text{ for all } N_* \in \mathcal{R}(G) \text{ and } j \in \{0,\dots,n\} \right\}$$

and

$$\mathbf{H}_\infty(\mathbb{F}) := \bigcap_{n\in\mathbb{N}} \mathbf{H}_n(\mathbb{F}).$$

Note that by Proposition 1.28.4 we have that if $\widehat{b_n}(G, N_*; \mathbb{F}) = 0$, then it is given by a proper limit.

Corollary 3.2 (comparing $\mathbb{Q}$- and $\mathbb{F}_p$-Betti number growth)**.** *Let $n \in \mathbb{N}$. Given a residually finite group G of type FP_n and a residual chain $N_* \in \mathcal{R}(G)$, we have that*

$$\widehat{b_n}(G, N_*; \mathbb{F}_p) \geq \widehat{b_n}(G, N_*; \mathbb{Q}).$$

Proof. By equation (14) we have for every $k \in \mathbb{N}$

$$\frac{b_n(N_k;\mathbb{F}_p)}{[G:N_k]} \geq \frac{b_n(N_k;\mathbb{Q})}{[G:N_k]}.$$

Passing to the limit superior yields the claim. □

Example 3.3 (zeroth Betti number growth)**.** Let G be a residually finite group and let $\mathbb{F}$ be a field. Then, for every residual chain $N_* \in \mathcal{R}(G)$, Example 2.50 implies that

$$\widehat{b_0}\,(G, N_*;\mathbb{F}) = \limsup_{k\to\infty} \frac{b_0(N_k;\mathbb{F})}{[G:N_k]} = \limsup_{k\to\infty} \frac{1}{[G:N_k]} = \frac{1}{|G|}.$$

In particular, $\mathbf{H}_0(\mathbb{F})$ is the class of all infinite residually finite groups.

Example 3.4 (Betti number growth of finite groups)**.** Let G be a finite group and let $\mathbb{F}$ be a field. By Example 1.37, we have for every residual chain $N_* \in \mathcal{R}(G)$ and every degree $n \in \mathbb{N}_{\geq 1}$:

$$\widehat{b_n}\,(G, N_*;\mathbb{F}) = \limsup_{k\to\infty} \frac{b_n(1;\mathbb{F})}{[G:1]} = 0.$$

In particular, all finite groups lie in $\mathbf{H}_\infty(\mathbb{F}) \setminus \mathbf{H}_0(\mathbb{F})$.

Example 3.5 (Betti number growth of free Abelian groups)**.** Let $G = \mathbb{Z}^m$ for $m \in \mathbb{N}_{\geq 1}$ and let $\mathbb{F}$ be a field. By Examples 1.38 and 2.14, for every residual chain N_* and every degree $n \in \mathbb{N}_{\geq 1}$ we obtain that

$$\widehat{b_n}\,(\mathbb{Z}^m, N_*;\mathbb{F}) = \limsup_{k\to\infty} \frac{b_n(\mathbb{Z}^m;\mathbb{F})}{[G:\mathbb{Z}^m]} = 0.$$

In particular, all free Abelian groups lie in $\mathbf{H}_\infty(\mathbb{F})$.

Example 3.6 (Betti number growth of free groups)**.** Let $G = F_m$ be a free group of rank $m \in \mathbb{N}_{\geq 1}$. We have seen in Example 1.39 that every finite index normal subgroup of G is itself a finitely generated free group. For every residual chain $N_* \in \mathcal{R}(G)$ we have

$$\widehat{b_1}\,(F_m, N_*;\mathbb{F}) = \limsup_{k\to\infty} \frac{(m-1)\cdot[F_m:N_k]+1}{[F_m:N_k]} = m-1.$$

where we have used our computation of Betti numbers of free groups in Example 2.51. Since group homology of free groups vanishes above degree 1, so does the higher Betti number growth and we get for $n \in \mathbb{N}_{\geq 1}$

$$\widehat{b_n}\,(F_m, N_*;\mathbb{F}) = \begin{cases} m-1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

In particular, $\widehat{b_n}\,(F_m)$ is a chain-independent proper limit for all $n \in \mathbb{N}$.

Example 3.7 (Betti number growth of orientable-surface groups)**.** Let $g \in \mathbb{N}_{\geq 1}$ and let $G = \pi_1(\Sigma_g)$ be an orientable-surface group of genus g. Let H be a finite index subgroup of G. Then H corresponds to a finite covering of Σ_g and is thus a closed orientable surface $\Sigma_{g'}$ itself. Its genus g' is determined by the Euler characteristic: using the proportionality principle 2.43 we deduce that

$$2-2g' = \chi(\Sigma_{g'}) = [G : H]\cdot\chi(\Sigma_g) = [G : H]\cdot(2-2g) \implies g' = [G : H]\cdot(g-1)+1.$$

By the computation of Betti numbers of orientable-surface groups in Example 2.53 we get for $n = 1$

$$\begin{aligned}
\widehat{b_1}\,(G, N_*; \mathbb{F}) &= \limsup_{k\to\infty} \frac{b_1(N_k; \mathbb{F})}{[G : N_k]} \\
&= \limsup_{k\to\infty} \frac{b_1(\pi_1(\Sigma_{g'}); \mathbb{F})}{[G : N_k]} \\
&= \limsup_{k\to\infty} \frac{2\big([G : N_k]\cdot(g-1)+1\big)}{[G : N_k]} \\
&= 2(g-1),
\end{aligned}$$

and $\widehat{b_n}\,(G, N_*; \mathbb{F}) = 0$ for $n \geq 2$. In particular $\widehat{b_n}\,(\pi_1(\Sigma_g); \mathbb{F})$ is a chain-independent proper limit.

Example 3.8 (Betti number growth of non-orientable-surface groups)**.** Let $g \in \mathbb{N}_{\geq 1}$ and let $G = \pi_1(S_g)$ be a non-orientable-surface group of genus g. Let H be a finite index subgroup of G with $d := [G : H]$. Then H corresponds to a finite covering of S_g and is thus a closed surface T itself; however, it can be both orientable and non-orientable. We enter a case distinction:

- If $T = \Sigma_{g'}$ is orientable, then one has

  $$2-2g' = \chi(\Sigma_{g'}) = d\cdot\chi(S_g) = d\cdot(2-g) \implies g' = \frac{1}{2}\big(d(g-2)+2\big).$$

  In particular, the first Betti number is given by

  $$b_1(\Sigma_{g'}; \mathbb{Q}) = 2g' = d(g-2)+2.$$

- If $T = S_{g'}$ is non-orientable, then one has

  $$2-g' = \chi(S_{g'}) = [G : H]\cdot\chi(S_g) = d\cdot(2-g) \implies g' = d(g-2)+2.$$

  In particular, the first Betti number is given by

  $$b_1(S_{g'}; \mathbb{Q}) = g'-1 = d(g-2)+1.$$

The key now is to see that in both cases, the asymptotic growths in the index coincide. Let $N_* \in \mathcal{R}(\pi_1(S_g))$ be a residual chain. Define a function

$$\tau : \mathbb{N} \to \{0,1\}, \tau(k) = \begin{cases} 0 & \text{if } N_k \text{ corresponds to an orientable surface,} \\ 1 & \text{if } N_k \text{ corresponds to an non-orientable surface,} \end{cases}$$

Then we compute the growth of the first Betti number as

$$\begin{aligned} \widehat{b_1}\,(\pi_1(S_g), N_*; \mathbb{F}) &= \limsup_{k\to\infty} \frac{b_1(N_k;\mathbb{F})}{[G : N_k]} \\ &= \limsup_{k\to\infty} \frac{[G : N_k]\cdot(g-2)+2-\tau(k)}{[G : N_k]} \\ &= (g-2) + \limsup_{k\to\infty} \frac{2-\tau(k)}{[G : N_k]} \\ &= g-2, \end{aligned}$$

and we have that $\widehat{b_n}\,(\pi_1(S_g), N_*; \mathbb{F}) = 0$ for $n \geq 2$. In particular, $\widehat{b_n}\,(\pi_1(S_g); \mathbb{F})$ is a chain-independent proper limit.

Question 3.9 (Betti number growth and coefficient field)**.** *Let p be prime. Does there exist a residually finite group G together with a residual chain $N_* \in \mathcal{R}(G)$ such that*

$$\widehat{b_1}\,(G, N_*; \mathbb{F}_p) > \widehat{b_1}\,(G, N_*; \mathbb{Q})?$$

The positive answer for degrees ≥ 2 was given by Avramidi–Okun–Schreve in 2021 by constructing a *right-angled Artin group* with (Example 3.59)

$$\widehat{b_2}\,(G, N_*; \mathbb{F}_2) = 1 \quad \text{and} \quad \widehat{b_2}\,(G, N_*; \mathbb{Q}) = 0.$$

Outlook 3.10 ($\mathbb{Q}$-Betti number growth)**.** In Chapter 3.1.6 we will see that the $\mathbb{Q}$-Betti number growth of groups with suitable finiteness type have extraordinary properties, expressed by Lück's approximation theorem.

Outlook 3.11 (Betti number growth of right-angled Artin groups)**.** In Chapter 3.2, we will see that Betti number growth has been computed for all right-angled Artin groups, and is related to the topology of their underlying flag complexes (see Theorem 3.57).

3.1.2. Torsion growth

Torsion growth is the growth of the logarithm of the cardinality of the torsion subgroup of the integral homology group.

Definition 3.12 (torsion growth)**.** Let G be a residually finite group, and let N_* be a residual chain in G. Let $n \in \mathbb{N}$. Then the *n-th (logarithmic) torsion (homology) growth of G with respect to N_** is defined as

$$\widehat{t_n}\,(G, N_*) := \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_n(N_k;\mathbb{Z})}{[G:N_k]}.$$

We introduce the notation

$$\mathbf{T}_n := \big\{G \text{ res. fin.} \bigm| \widehat{t_j}\,(G,N_*) = 0 \text{ for all } N_* \in \mathcal{R}(G) \text{ and } j \in \{0,\dots,n\}\big\}$$

and

$$\mathbf{T}_\infty := \bigcap_{n\in\mathbb{N}} \mathbf{T}_n.$$

Note that by Proposition 1.28.4 we have that if $\widehat{t_n}\,(G,N_*) = 0$, then it is given by a proper limit.

Example 3.13 (zeroth torsion growth)**.** Let G be a residually finite group. Then, for every residual chain $N_* \in \mathcal{R}(G)$ we have by Example 2.12

$$\widehat{t_0}\,(G,N_*) = \limsup_{k\to\infty} \frac{\log \operatorname{tors} \mathbb{Z}}{[G:N_k]} = 0.$$

In particular, $\mathbf{T}_0$ is the class of all residually finite groups.

Example 3.14 (torsion growth of finite groups)**.** Let G be a finite group. By Example 1.37, we have for every residual chain $N_* \in \mathcal{R}(G)$ and every degree $n \in \mathbb{N}$:

$$\widehat{t_n}\,(G,N_*) = \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_n(1)}{[G:1]} = 0.$$

In particular, all finite groups lie in $\mathbf{T}_\infty$.

Example 3.15 (torsion growth of free Abelian groups)**.** Let $G = \mathbb{Z}^m$ for $m \in \mathbb{N}_{\geq 1}$. By Examples 1.38 and 2.14, for every residual chain N_* and every degree $n \in \mathbb{N}$ we obtain that

$$\widehat{t_n}\,(\mathbb{Z}^m, N_*) = 0.$$

In particular, all free Abelian groups lie in $\mathbf{T}_\infty$.

Example 3.16 (torsion growth of free groups)**.** Let $G = F_m$ be a free group of rank $m \in \mathbb{N}_{\geq 1}$. We have seen in Example 1.39 that every finite index normal subgroup of G is itself a finitely generated free group. In particular, for every residual chain N_* and every degree $n \in \mathbb{N}$ we obtain that

$$\widehat{t_n}\,(F_m, N_*) = \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_n(N_k;\mathbb{Z})}{[F_m:N_k]} = \limsup_{k\to\infty} \frac{0}{[F_m:N_k]} = 0,$$

where we have used our computation of group homology of free groups in Example 2.15. In particular, all free groups lie in $\mathbf{T}_\infty$.

Example 3.17 (torsion growth of surface groups)**.** Let G be a surface group. We have seen in Example 1.40 that every subgroup of a surface group is a surface group. It follows from the computation of group homology in Example 2.17 that $\operatorname{tors} H_n(G;\mathbb{Z}) \leq 2$ for any (both orientable and non-orientable) surface group G. Therefore

$$0 \leq \widehat{t}_n\,(G, N_*) = \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_n(N_k;\mathbb{Z})}{[G:N_k]} \leq \limsup_{k\to\infty} \frac{\log 2}{[G:N_k]} = 0.$$

In particular, all surface groups lie in $\mathbf{T}_\infty$.

Question 3.18 (positive value for torsion growth)**.** *For $n \in \mathbb{N}_{\geq 1}$, does there exist a (finitely presented) residually finite group that does* not *lie in $\mathbf{T}_n$?*

The positive answer for degrees ≥ 2 was given by Avramidi–Okun–Schreve in 2021 by constructing a *right-angled Artin group* with positive torsion growth bounded below by $\log 2$ (Example 3.59). This lower bound was promoted to a sharp equality by Okun–Schreve in 2024 (see Example 3.62). The situation in degree 1 will be discussed in Chapter 3.1.4.

Outlook 3.19 (torsion growth of right-angled Artin groups)**.** In Chapter 3.2, we will see that torsion growth has been computed for all right-angled Artin groups, and is related to the topology of their underlying flag complexes (see Theorem 3.60).

3.1.3. Betti number vs. torsion growth

A group with Betti number growth strictly depending on the characteristic of the coefficient field (answering Question 3.9) necessarily has positive torsion growth in the same degree or one degree lower (answering Question 3.18). This essentially follows from equation (14), which relates p-torsion in the homology groups to Betti numbers with coefficients $\mathbb{Q}$ and $\mathbb{F}_p$.

Theorem 3.20 (torsion growth and Betti number growth)**.** *Let G be a residually finite group of type FP_{n+1}, together with a residual chain $N_* \in \mathcal{R}(G)$. Then we have for every prime p and every $n \in \mathbb{N}_{\geq 1}$*

$$\widehat{t}_n\,(G, N_*) + \widehat{t}_{n-1}\,(G, N_*) \geq \Big(\widehat{b}_n\,(G, N_*;\mathbb{F}_p) - \widehat{b}_n\,(G, N_*;\mathbb{Q})\Big) \cdot \log(p).$$

Proof. By Theorem 3.1.4, we have that the Betti number growths on the right side are bounded, so that we do not run into the indeterminate form $\infty - \infty$. Notice that for every finitely generated $\mathbb{Z}$-module A with decomposition $A \cong_{\mathbb{Z}} \mathbb{Z}^d \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i}$ one has that

$$\log \operatorname{tors} A = \log\Big(\prod_{i\in\mathcal{I}} p_i^{n_i}\Big) = \sum_{i\in\mathcal{I}} n_i \cdot \log(p_i) \geq \max_{p \text{ prime}} \big\{ \operatorname{mult}_p A \cdot \log(p) \big\}$$

Fix $k \in \mathbb{N}$ and a prime p. Use equation (14) to obtain

$$\begin{aligned}
&\big(b_n(N_k;\mathbb{F}_p) - b_n(N_k;\mathbb{Q})\big) \cdot \log(p)\\
=\;& \operatorname{mult}_p H_n(N_k;\mathbb{Z}) \cdot \log(p) + \operatorname{mult}_p H_{n-1}(N_k;\mathbb{Z}) \cdot \log(p)\\
\le\;& \log \operatorname{tors} H_n(N_k;\mathbb{Z}) + \log \operatorname{tors} H_{n-1}(N_k;\mathbb{Z}).
\end{aligned}$$

Divide by $[G : N_k]$ and pass to the limit superior

$$\begin{aligned}
&\left(\widehat{b}_n\,(G, N_*;\mathbb{F}_p) - \widehat{b}_n\,(G, N_*;\mathbb{Q})\,\right) \cdot \log(p)\\
=\;& \limsup_{k\to\infty} \left(\frac{b_n(N_k;\mathbb{F}_p)\cdot \log(p)}{[G:N_k]} - \frac{b_n(N_k;\mathbb{Q})\cdot \log(p)}{[G:N_k]}\right)\\
\le\;& \limsup_{k\to\infty} \left(\frac{\log \operatorname{tors} H_n(N_k;\mathbb{Z})}{[G:N_k]} + \frac{\log \operatorname{tors} H_{n-1}(N_k;\mathbb{Z})}{[G:N_k]}\right)\\
\le\;& \widehat{t}_n\,(G, N_*) + \widehat{t}_{n-1}\,(G, N_*)\,.
\end{aligned}$$

In the first line, to obtain equality instead of only "$\geq$", we use the fact that $b_n^{(2)}(G, N_*;\mathbb{Q})$ is given by a proper limit. This is a consequence of Lück's approximation theorem 3.34, that we will encounter soon. □

Corollary 3.21 (positive torsion growth via Betti numbers)**.** *Let $n \in \mathbb{N}$ and let p be prime. Let G be a residually finite group of type FP_{n+1} together with a residual chain $N_* \in \mathcal{R}(G)$ such that*

$$\widehat{b}_n\,(G, N_*, \mathbb{Q}) \neq \widehat{b}_n\,(G, N_*, \mathbb{F}_p)\,.$$

Then $\max\left\{\widehat{t}_n\,(G, N_*)\,, \widehat{t}_{n-1}\,(G, N_*)\,\right\} > 0.$

Proof. This follows immediately from Theorem 3.20 and Corollary 3.2. □

3.1.4. Bounding homology growth

In Chapter 2.2.3, we have seen estimates that control the "size" of group homology of finite index subgroups. This implies that torsion growth and Betti number growth of groups with suitable finiteness type are finite real numbers. However, Kar–Kropholler–Nikolov showed that the finiteness assumption is necessary in degree 1.

Corollary 3.22 (bounded growth in the presence of FP_n)**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G be a residually finite group of type FP_n. Then, for all residual chains $N_* \in \mathcal{R}(G)$:*

$$\begin{aligned}
&\widehat{b}_j\,(G, N_*;\mathbb{F}) < \infty \text{ for all } j \in \{0, \dots, n\},\\
\textit{and}\qquad &\widehat{t}_j\,(G, N_*) < \infty \text{ for all } j \in \{0, \dots, n-1\}.
\end{aligned}$$

Proof. By Theorems 2.55 and 2.60 it follows that

$$\widehat{b}_j\,(G, N_*; \mathbb{F}) = \limsup_{k\to\infty} \frac{b_j(N_k; \mathbb{F})}{[G : N_k]} \leq \limsup_{k\to\infty} \frac{c_j(G) \cdot [G : N_k]}{[G : N_k]} \leq c_j(G) < \infty$$

for all $j \in \{0, \dots, n\}$; and

$$\widehat{t}_j\,(G, N_*) = \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_j(N_k; \mathbb{Z})}{[G : N_k]} \leq \limsup_{k\to\infty} \frac{c_j(G) \cdot [G : N_k]}{[G : N_k]} \leq c_j(G) < \infty$$

for all $j \in \{0, \dots, n-1\}$. □

As a special instance of the more general Theorem 2.60, Abért, Gelander and Nikolov have shown the following bound in the case of finitely presented (type F_2) groups.

Theorem 3.23 (torsion in abelianization [AGN17, Lemma 27])**.** *Let G be a group with finite presentation $\langle X \mid R\rangle$. Let b be the maximal word length of the relations in R. Let $H \subseteq G$ be a finite index subgroup. Then*

$$\widehat{t}_1\,(G) \leq |X| \cdot \log(b).$$

For $n = 1$, Kar, Kropholler and Nikolov show that the assumption that G is of type FP_2 in Corollary 3.22 is necessary.

Theorem 3.24 (fast growth of torsion in H_1 [KKN17, Theorem 3])**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be any function. There exists a finitely generated residually finite group G, together with a residual chain $N_* \in \mathcal{R}(G)$ such that for all $k \in \mathbb{N}$:*

$$\operatorname{tors}\big(H_1(N_k; \mathbb{Z})\big) > f\big([G : N_k]\big).$$

Corollary 3.25 (unbounded first torsion growth)**.** *For every $C \in \mathbb{N}$ there exists a finitely generated residually finite group G, together with a residual chain $N_* \in \mathcal{R}(G)$ such that $\widehat{t}_1\,(G, N_*) > C$. Moreover, there exists a finitely generated residually finite group G, together with a residual chain $N_* \in \mathcal{R}(G)$ such that $\widehat{t}_1\,(G, N_*) = \infty$.*

Proof. Given $C \in \mathbb{N}$, consider the function

$$\begin{aligned} f_C : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \lceil \exp(C \cdot n) \rceil. \end{aligned}$$

Then by Theorem 3.24 there exists a finitely generated residually finite group G, together with a residual chain $N_* \in \mathcal{R}(G)$ such that

$$\begin{aligned}
\widehat{t_1}\,(G, N_*) &= \limsup_{k\to\infty} \frac{\log \operatorname{tors} H_1(N_k;\mathbb{Z})}{[G:N_k]} \\
&\geq \limsup_{k\to\infty} \frac{\log\Big(f_C\big([G:N_k]\big)\Big)}{[G:N_k]} \\
&= \limsup_{k\to\infty} \frac{\log\Big(\big\lceil \exp(C\cdot [G:N_k])\big\rceil\Big)}{[G:N_k]} \\
&\geq \limsup_{k\to\infty} \frac{C\cdot [G:N_k]}{[G:N_k]} \\
&= C.
\end{aligned}$$

For the second part, consider $f_\infty(n) := \lceil \exp(n^2)\rceil$. Again, by Theorem 3.24 there exists a finitely generated residually finite group G, together with a residual chain $N_* \in \mathcal{R}(G)$ such that

$$\begin{aligned}
\widehat{t_1}\,(G, N_*) &= \limsup_{k\to\infty} \frac{\log \operatorname{tors}\big(H_1(N_k;\mathbb{Z})\big)}{[G:N_k]} \\
&\geq \limsup_{k\to\infty} \frac{\log\Big(f_\infty\big([G:N_k]\big)\Big)}{[G:N_k]} \\
&= \limsup_{k\to\infty} \frac{\log\Big(\big\lceil \exp([G:N_k]^2)\big\rceil\Big)}{[G:N_k]} \\
&\geq \limsup_{k\to\infty} \frac{[G:N_k]^2}{[G:N_k]} \\
&= \infty. \qquad \square
\end{aligned}$$

The groups constructed in Corollary 3.25 are not finitely presented. In fact, the following is an open question.

Question 3.26 (positive first torsion growth [Nik21, Question 1])**.** *Is there a finitely presented residually finite group G with a residual chain $N_* \in \mathcal{R}(G)$, such that $\widehat{t_1}\,(G, N_*) > 0$?*

Notice that by Corollary 3.21, a positive answer to the above question would be implied by Question 3.9.

Remark 3.27 (cheap rebuilding property)**.** In a relatively recent breakthrough paper by Abért, Bergeron, Frączyk and Gaboriau, the authors introduce a sequence of classes of residually finite groups satisfying the *cheap $*$-rebuilding property* which is degreewise contained in $\mathbf{H}_*(\mathbb{F})$ and $\mathbf{T}_{*-1}$ [ABFG24]. The definition "requires the existence of models for classifying spaces satisfying delicate estimates on the number of cells and norms of boundary maps, homotopy equivalences, and homotopies" [LLMSU24], similarly to what we have seen in Example 2.63. Moreover, this construction comes with a handy *Bootstrapping theorem*, which allows to establish the cheap rebuilding property of a group by exhibiting a sufficiently nice free action on a CW-complex [ABFG24, Theorem F]. These results can be applied to show vanishing of homology growth for many classes of groups, including special linear groups, mapping class groups, certain Artin groups [ABFG24], outer automorphism groups of free products of $\mathbb{Z}/2$ [GGH24], mapping tori of polynomially growing automorphisms [AHK24][AGHK25], and inner-amenable groups [Usc24]. Li, Löh, Moraschini, Sauer and Uschold have introduced an algebraic counterpart. Though it is heavily inspired by the geometric version, it is not clear whether there are implications [LLMSU24, Remarks 2.11, 4.25].

3.1.5. Free products

Our goal is to apply Corollary 1.53 to torsion growth and Betti number growth. In order to do this, we first need to understand how group homology behaves with respect to free products.

Theorem 3.28 (Mayer–Vietoris [Bro82, Corollary II.7.7])**.** *Let G_1 and G_2 be groups. Let G_0 be a group with embeddings $i_1 : G_0 \hookrightarrow G_1$ and $i_2 : G_0 \hookrightarrow G_2$. There exists a long exact sequence of $\mathbb{Z}$-modules*

$$\begin{aligned}\dots \to H_n(G_0) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G_1 *_{G_0} G_2) \to H_{n-1}(G_0) \to \dots \\ \dots \to H_0(G_1) \oplus H_0(G_2) \to H_0(G_1 *_{G_0} G_2) \to 0.\end{aligned}$$

Corollary 3.29 (group homology of free products)**.** *Given two groups G_1 and G_2 we have that for all $n \in \mathbb{N}_{\geq 1}$*

$$H_n(G_1 * G_2; \mathbb{Z}) \cong_{\mathbb{Z}} H_n(G_1; \mathbb{Z}) \oplus H_n(G_2; \mathbb{Z}).$$

Given a field $\mathbb{F}$ we also get

$$H_n(G_1 * G_2; \mathbb{F}) \cong_{\mathbb{F}} H_n(G_1; \mathbb{F}) \oplus H_n(G_2; \mathbb{F}).$$

Proof. Free products are amalgamated products over the trivial subgroups, so by Example 2.13, the exact sequence from Theorem 3.29 becomes for $n \geq 1$

$$\dots \to 0 \to H_n(G_1) \oplus H_n(G_2) \to H_n(G_1 * G_2) \to 0 \to \dots,$$

inducing isomorphisms as claimed. The second claim with field coefficients follows from the integral case by using the isomorphism (12) from the universal coefficient theorem 2.47 together with the fact that $\operatorname{Tor}_1^{\mathbb{Z}}$ and $\otimes_{\mathbb{Z}}\mathbb{F}$ commute with direct sums. □

Corollary 3.30 (torsion growth of free products)**.** *Let $n \in \mathbb{N}$, let G_1 and G_2 be residually finite groups. Then for every residual chain $N_* \in \mathcal{R}(G_1 * G_2)$ with associated residual chains $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ as constructed in Proposition 1.17, we have*

$$\widehat{t}_n\left(G_1 * G_2, N_*\right) \leq \widehat{t}_n\left(G_1, N_*^1\right) + \widehat{t}_n\left(G_2, N_*^2\right),$$

with equality if $\widehat{t}_n\left(G_1, N_*^1\right)$ is a proper limit.

Proof. For $n = 0$, the claim trivially holds by Example 3.13. For $n \in \mathbb{N}_{\geq 1}$, given two groups G_1 and G_2, the Mayer–Vietoris decomposition in Corollary 3.29 yields

$$\begin{aligned}
\log \operatorname{tors} H_n(G_1 * G_2; \mathbb{Z}) &= \log \operatorname{tors}\big(H_n(G_1;\mathbb{Z}) \oplus H_n(G_2;\mathbb{Z})\big)\\
&= \log\big(\operatorname{tors} H_n(G_1;\mathbb{Z}) \cdot \operatorname{tors} H_n(G_2;\mathbb{Z})\big)\\
&= \log \operatorname{tors} H_n(G_1;\mathbb{Z}) + \log \operatorname{tors} H_n(G_2;\mathbb{Z}).
\end{aligned}$$

Moreover, we have that $\log \operatorname{tors} H_n(\mathbb{Z};\mathbb{Z}) = 0$. The claim now follows from Corollary 1.53.1. □

Corollary 3.31 (Betti number growth of free products)**.** *Let $\mathbb{F}$ be a field. Let G_1 and G_2 be residually finite groups. For every residual chain $N_* \in \mathcal{R}(G_1 * G_2)$ with associated residual chains $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ as constructed in Proposition 1.17, we have*

$$\widehat{b}_1\left(G_1 * G_2, N_*; \mathbb{F}\right) \leq \widehat{b}_1\left(G_1, N_*^1; \mathbb{F}\right) + \widehat{b}_1\left(G_2, N_*^2; \mathbb{F}\right) + 1 - \frac{1}{|G_1|} - \frac{1}{|G_2|} + \frac{1}{|G_1 * G_2|},$$

with equality if $\widehat{b}_1\left(G_1, N_*^1; \mathbb{F}\right)$ is a proper limit. For $n \in \mathbb{N}_{\geq 2}$ we have

$$\widehat{b}_n\left(G_1 * G_2, N_*; \mathbb{F}\right) \leq \widehat{b}_n\left(G_1, N_*^1; \mathbb{F}\right) + \widehat{b}_n\left(G_2, N_*^2; \mathbb{F}\right),$$

with equality if $\widehat{b}_n\left(G_1, N_*^1; \mathbb{F}\right)$ is a proper limit.

Proof. By Corollary 3.29 we have for all groups G_1 and G_2 and $n \in \mathbb{N}_{\geq 1}$

$$\begin{aligned}
b_n(G_1 * G_2; \mathbb{F}) &= \dim_{\mathbb{F}} H_n(G_1 * G_2; \mathbb{F})\\
&= \dim_{\mathbb{F}}\big(H_n(G_1;\mathbb{F}) \oplus H_n(G_2;\mathbb{F})\big)\\
&= \dim_{\mathbb{F}} H_n(G_1;\mathbb{F}) + \dim_{\mathbb{F}} H_n(G_2;\mathbb{F})\\
&= b_n(G_1;\mathbb{F}) + b_n(G_2;\mathbb{F}).
\end{aligned}$$

Moreover, we have that for $n \in \mathbb{N}_{\geq 1}$ (Example 2.51)

$$b_n(\mathbb{Z};\mathbb{F}) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

The two claims now follow from the two parts of Corollary 1.53. $\square$

Corollary 3.32 (vanishing of homology growth of free products)**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field and let G_1 and G_2 be residually finite groups.*

1. *If $G_1, G_2 \in \mathbf{H}_n(\mathbb{F})$, then $G_1 * G_2 \in \mathbf{H}_n(\mathbb{F})$.*
2. *If $G_1, G_2 \in \mathbf{T}_n$, then $G_1 * G_2 \in \mathbf{T}_n$.*

Proof. Let $N_* \in \mathcal{R}(G_1 * G_2)$ be a residual chain, and let $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ be the associated chains as constructed in Proposition 1.17.
Ad 1. Note that by assumption, G_1 and G_2 are necessarily infinite. By Corollary 3.31, for all $m \in \{0, \dots, n\}$

$$\widehat{b}_m\left(G_1 * G_2, N_*;\mathbb{F}\right) \leq \widehat{b}_m\left(G_1, N_*^1;\mathbb{F}\right) + \widehat{b}_m\left(G_2, N_*^2;\mathbb{F}\right) = 0.$$

Ad 2. By Corollary 3.30, for all $m \in \{0, \dots, n\}$

$$\widehat{t}_m\left(G_1 * G_2, N_*\right) \leq \widehat{t}_m\left(G_1, N_*^1\right) + \widehat{t}_m\left(G_2, N_*^2\right) = 0. \qquad \square$$

Remark 3.33. Corollary 3.32.1 recovers a result of Li–Löh–Moraschini–Sauer–Uschold, using that $G_1 * G_2$ is the fundamental group of a graph of groups with trivial edge group [LLMSU24, Corollary 3.7.(i)]. As $\mathbf{T}_*$ is not known to be a bootstrappable property, their result does not apply to Corollary 3.32.2.

3.1.6. Lück's approximation theorem

Recall that in Chapter 1.2 we have formulated Lück's *approximation problem* 1.33 inherent to the definition of growth of invariants:

1. Does the sequence defining $\widehat{I}(G, N_*)$ converge?
2. Does $\widehat{I}(G, N_*)$ depend on the chain $N_* \in \mathcal{R}(G)$?
3. Does $\widehat{I}$ admit a different interpretation?

In most cases, answers to these questions are not known. A beacon of hope is *Lück's approximation theorem*, which arguably sparked the interest for homology growth in the first place.

Theorem 3.34 (Lück's approximation theorem [Lüc94]). *Let $n \in \mathbb{N}$, and let G be a residually finite group of type* FP_{n+1}. *Then for every residual chain $N_* \in \mathcal{R}(G)$ one has*

$$\widehat{b}_n\,(G, N_*; \mathbb{Q}) = b_n^{(2)}(G),$$

where $b_n^{(2)}(G)$ is the n-th L^2-Betti number of G. *In particular, $\widehat{b}_n\,(G; \mathbb{Q})$ is a chain-independent proper limit.*

Remark 3.35 (L^2-Betti numbers). The L^2-*Betti numbers* of a group G are an equivariant version of ordinary Betti numbers. Since there is no nice dimension theory over $\mathbb{Z}G$, one passes to L^2-completions of the group ring and the *von Neumann algebra.* This allows for a suitable notion of *von Neumann dimension.* One then defines the L^2-Betti numbers of G as the von Neumann dimensions of the homology groups of the L^2-completion of the cellular chain complex of $\widetilde{X}$, where $\widetilde{X}$ is the universal covering of a classifying space X of G equipped with the structure of a free G-CW-complex [LU23, Chapter 2.1][Kam19, Chapter 3.3]. In this thesis, we can not do the theory of L^2-invariants any justice. For details, we refer to the books by Lück [Lüc02] and Kammeyer [Kam19].

Remark 3.36 (necessity of the assumptions). The assumption in Lück's approximation theorem that G is of type FP_{n+1} is necessary for $n = 1$:

- Osin showed that if the statement of Lück's approximation theorem were true in degree $n = 1$ for all finitely generated (i.e., type FP_1) groups, this would imply the existence of a non-residually finite hyperbolic group (!) [Osi09].
- However, there does exists a finitely generated residually finite group G together with a residual chain $N_* \in \mathcal{R}(G)$ such that $b_1^{(2)}(G) > 0$, but $\widehat{b}_1\,(G, N_*; \mathbb{Q}) = 0$ for all chains [LO11, Theorem 1.2].
- There exists a finitely generated residually finite group G and a residual chain $N_* \in \mathcal{R}(G)$ such that $\widehat{b}_n\,(G, N_*; \mathbb{Q})$ is *not* a proper limit [EL14, Theorem 1.3].

Example 3.37 (computing L^2-Betti numbers). Our computations of $\mathbb{Q}$-Betti number growth of groups in Chapter 3.1.1 can be translated via Lück's approximation theorem into computations of L^2-Betti numbers (though these could be obtained directly from the definitions):

- If G is a finitely generated residually finite group, then $b_0^{(2)}(G) = |G|^{-1}$.
- If G is finite, then $b_n^{(2)}(G) = 0$ for all $n \in \mathbb{N}_{\geq 1}$.
- If $G = \mathbb{Z}^m$ for $m \in \mathbb{N}_{\geq 1}$, then $b_n^{(2)}(\mathbb{Z}^m) = 0$ for all $n \in \mathbb{N}_{\geq 1}$.

- If $G = F_m$ for $m \in \mathbb{N}_{\geq 1}$, then $b_1^{(2)}(F_m) = m - 1$.
- If G is a surface group of genus $g \in \mathbb{N}_{\geq 1}$, then

$$b_1^{(2)}(G) = \begin{cases} 2(g-1) & \text{if } G \text{ is an orientable-surface group,} \\ g-2 & \text{if } G \text{ is an non-orientable-surface group.} \end{cases}$$

Remarkably, eventhough L^2-Betti numbers differ fundamentally from ordinary Betti numbers, for groups of type F, the alternating sum still recovers the Euler characteristic:

Theorem 3.38 (Euler–Poincaré formula)**.** *Let G be a group of type* F*. Then*

$$\sum_{n\in\mathbb{N}} (-1)^n \cdot b_n^{(2)}(G) = \chi(G).$$

Proof. A proper proof using the definition of L^2-Betti numbers and properties of the von Neumann dimension can be found in Lück's monograph [Lüc02, Theorem 1.35(2)]. Instead, we "re-prove" this theorem for a residually finite group G, from the perspective of Betti number growth. Let $N_* \in \mathcal{R}(G)$ be an arbitrary residual chain in G. Using the proportionality principle and Lück's approximation theorem (G is of type F) we get that

$$\chi(G) = \frac{\chi(N_k)}{[G : N_k]} = \sum_{n\in\mathbb{N}} (-1)^n \cdot \frac{b_n(N_k; \mathbb{Q})}{[G : N_k]} \xrightarrow{k\to\infty} \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n^{(2)}(G). \qquad \square$$

Using our axiomatic work on growth of invariants of free products and Lück's approximation theorem, we recover the following well-known result about free products of L^2-Betti numbers, assuming a suitable finiteness type [Kam19, Theorem 4.15(ii)]:

Corollary 3.39 (L^2-Betti numbers of free products)**.** *Let $n \in \mathbb{N}_{\geq 2}$, and let G_1 and G_2 be two non-trivial residually finite groups of type* FP_{n+1}*. Then we have for all $j \in \{0, \dots, n\}$:*

$$b_j^{(2)}(G_1 * G_2) = \begin{cases} b_1^{(2)}(G_1) + b_1^{(2)}(G_2) + 1 - \dfrac{1}{|G_1|} - \dfrac{1}{|G_2|} & \textit{if } j = 1, \\ b_j^{(2)}(G_1) + b_j^{(2)}(G_2) & \textit{if } j \in \{2, \dots, n\}. \end{cases}$$

Proof. This follows immediately from Corollary 3.31 and Lück's approximation theorem 3.34. $\square$

Earlier we have introduced the rank gradient and deficiency gradient as first examples of growth of invariants whose definitions did not require preliminaries on group homology. In reality, they are very much related to Betti number growth and thus – via Lück's approximation theorem – to L^2-Betti numbers.

Corollary 3.40 (rank gradient and first L^2-Betti number)**.** *Let G be a finitely presented residually finite group. Then*

$$\mathrm{RG}(G) \geq b_1^{(2)}(G).$$

Proof. If G is finite, then the statement follows from Examples 3.4 and 1.64.

Assume that G is infinite. We show that for every residual chain $N_* \in \mathcal{R}(G)$, one has that $\mathrm{RG}(G, N_*) = \widehat{d}(G, N_*) \geq b_1^{(2)}(G)$. For this, by Lück's approximation theorem 3.34 it suffices to show that

$$d(H) \geq b_1(H; \mathbb{Q}),$$

for every finitely presented group H. This follows from the fact that we can build a classifying space for H with 1-skeleton corresponding to the generators of H, see the proof sketch of Theorem 2.20. □

Question 3.41 ([KKL23, Questions 9.3 and 9.4])**.** *Kirstein, Kremer and Lück pose the following two questions:*

- *Let G be an infinite finitely generated residually finite group and let $N_* \in \mathcal{R}(G)$ be a residual chain. Do we have*

$$b_1^{(2)}(G) = \mathrm{cost}(G) - 1 = \mathrm{RG}(G, N_*)?$$

- *Let G be an infinite finitely presented residually finite group and let $N_* \in \mathcal{R}(G)$ be a residual chain. Let $\mathbb{F}$ be a field. Do we have that $\widehat{b_1}\,(G, N_*; \mathbb{F})$ is a proper limit and*

$$\widehat{b_1}\,(G, N_*; \mathbb{F}) = b_1^{(2)}(G) = \mathrm{cost}(G) - 1 = \mathrm{RG}(G, N_*)?$$

3.2. Homology growth of right-angled Artin groups

One of the difficulties of working with homology growth is the lack of rich, well-understood examples. Particularly, the study of torsion growth is marked by the existence of many vanishing results and virtually no constructions of non-vanishing values. So far, we have yet to encounter a finitely presented group with positive torsion growth, or at least a group where we know the exact value of the positive growth. Luckily, by the work of Avramidi, Okun and Schreve, there exists a class of groups where homology growth is very accessible: the *right-angled Artin groups*. This class interpolates between free Abelian groups and free groups, and often comes up as a source of counterexamples due to its connection to topology.

3.2.1. Artin groups

(Right-angled) Artin groups are groups associated to labelled simplicial graphs, whose presentation is governed by the graph structure. For a readable introduction to (right-angled) Artin groups we refer to Koberda's lecture notes [Kob13] and Charney's introductory article [Cha07].

Definition 3.42 (Artin group)**.** An *Artin group* A is a group that admits a presentation of the form

$$A = \Big\langle s_1, \dots, s_n \Big| \underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}} \text{ for all } i \neq j \Big\rangle, \tag{20}$$

where $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $i, j \in \{1, \dots, n\}$. If $m_{ij} = \infty$, we simply omit the corresponding relation in the presentation above.

We say that an Artin group A is *right-angled* (abbreviated as *RAAG*), if $m_{ij} \in \{2, \infty\}$ for all $i, j \in \{1, \dots, n\}$.

If we add the additional relations $s_i^2 = 1$ for all $i \in \{1, \dots, n\}$ to an Artin group, we obtain a *(right-angled) Coxeter group* W. We say that an Artin group is *of finite type* (or *spherical*), if the associated Coxeter group is finite, and *of infinite type* (or *aspherical*) otherwise. Notice that a right-angled Artin group is of finite type if and only if it is free Abelian.

Let $\Gamma = (V, E, l)$ be a finite simplicial graph, with every edge $e \in E$ labelled by a positive integer $l(e) \in \mathbb{N}_{\geq 2}$. The *associated Artin group* A_Γ is the Artin group given by the presentation

$$A_\Gamma = \Big\langle v \in V \Big| \underbrace{v_i v_j v_i \dots}_{l(e)} = \underbrace{v_j v_i v_j \dots}_{l(e)} \text{ for all edges } e = (v_i, v_j) \in E \Big\rangle.$$

Clearly, this gives rise to an Artin group. Conversely, given an Artin group with presentation as in (20), we can build an associated labelled finite simplicial graph Γ by setting $V := \{s_1, \dots, s_n\}$, such that $e = \{s_i, s_j\} \in E$ if and only if $m_{ij} < \infty$. In this case, e has label $l(e) := m_{ij}$.

The same way, one gets a correspondence between right-angled Artin groups and (unlabelled) finite simplicial graphs by performing the above construction with constant edge labels 2.

Example 3.43 (right-angled Artin groups)**.** Consider the finite simplicial graphs in Figure 2. For Γ_2, Γ_3, Γ_4, and Γ_5, we label the vertices in the same way as in Γ_1. In Γ_6, the vertices of the base square are also labeled as in Γ_1.

- $A_{\Gamma_1} = \langle a, b, c, d \mid \ \rangle \cong F_4$,

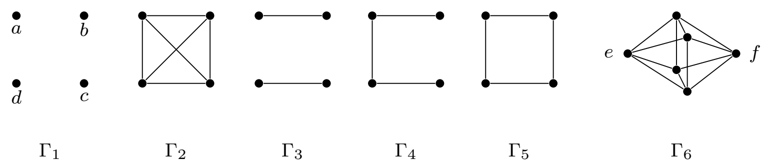


Figure 2: Some graphs

- $A_{\Gamma_2} = \big\langle a, b, c, d \;\big|\; [a, b], [a, c], [a, d], [b, c], [b, d], [c, d] \big\rangle \cong \mathbb{Z}^4$,
- $A_{\Gamma_3} = \big\langle a, b, c, d \;\big|\; [a, b], [c, d] \big\rangle \cong \mathbb{Z}^2\langle a, b\rangle * \mathbb{Z}^2\langle c, d\rangle$,
- $A_{\Gamma_4} = \big\langle a, b, c, d \;\big|\; [a, b], [a, d], [b, c] \big\rangle \cong \big(\mathbb{Z}\langle a\rangle \times F_2\langle b, d\rangle\big) *_{\mathbb{Z}\langle d\rangle} \mathbb{Z}^2\langle c, d\rangle$,
- $A_{\Gamma_5} = \big\langle a, b, c, d \;\big|\; [a, b], [a, d], [b, c], [c, d] \big\rangle \cong F_2\langle a, c\rangle \times F_2\langle b, d\rangle$,
- $A_{\Gamma_6} = \big\langle a, b, c, d, e, f \;\big|\; [a, b], [a, d], [a, e], [a, f], \dots, [d, f] \big\rangle \cong A_{\Gamma_5} \times F_2\langle e, f\rangle$.

By comparing the canonical presentations of right-angled Artin groups, amalgamated products and direct products, it is an easy to generalize the computations in the above examples.

Lemma 3.44 (free products of RAAGs)**.** *Let Γ be a finite simplicial graph and let Γ_1 and Γ_2 be two induced subgraphs. If $\Gamma = \Gamma_1 \cup \Gamma_2$ then*

$$A_\Gamma \cong A_{\Gamma_1} *_{A_{\Gamma_0}} A_{\Gamma_2},$$

where $\Gamma_0 := \Gamma_1 \cap \Gamma_2$. In particular: If $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then

$$A_\Gamma \cong A_{\Gamma_1} * A_{\Gamma_2}.$$

Lemma 3.45 (direct products of RAAGs)**.** *Let Γ be a finite simplicial graph and let Γ_1 and Γ_2 be two induced subgraphs with the following properties:*

- $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$,
- $V(\Gamma) = V(\Gamma_1) \sqcup V(\Gamma_2)$,
- $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2) \sqcup \big\{\{\gamma_1, \gamma_2\} \mid \gamma_1 \in V(\Gamma_1), \gamma_2 \in V(\Gamma_2)\big\}$.

Then there is a canonical isomorphism $A_\Gamma \cong A_{\Gamma_1} \times A_{\Gamma_2}$ induced by inclusion of generators.

Remark 3.46 (graph-theoretical join)**.** In the setting of Lemma 3.45, Γ is called a *graph-theoretical join of* Γ_1 *and* Γ_2. There exists a more general non-discrete version of the join construction (Definition 3.70). In the case of geometric simplicial complexes, this operation on the 1-skeleta corresponds precisely to the (geometric realization of) graph-theoretical join.

Definition 3.47 (Salvetti complex)**.** Let Γ be a finite simplicial graph. The *Salvetti complex* Sal_Γ *associated to* Γ is a finite CW-complex constructed as follows:

- the 0-skeleton is given by a single point;
- the 1-skeleton is given by a wedge of circles $\bigvee_{v\in V}\mathbb{S}^1$;
- given the $(n-1)$-skeleton for $n \in \mathbb{N}_{\geq 2}$, we construct the n-skeleton by glueing in n-tori for every set of n vertices spanning a complete subgraph of Γ along the corresponding relator.

Example 3.48 (group homology of right-angled Artin groups)**.** Given a finite simplicial graph Γ, the associated Salvetti complex Sal_Γ admits the structure of a finite CW-complex with fundamental group A_Γ. Its universal covering is a CAT(0) cube complex and thus contractible [Cha07, Theorem 2.6][CD95, Theorem 3.1.1]. In particular, Sal_Γ is a finite classifying space for A_Γ whose cells are in bijective correspondence to *cliques* in Γ: given $n \in \mathbb{N}_{\geq 1}$, an *$n$-clique in* Γ is a complete full subgraph on n vertices in Γ. We define the *n-clique number of* Γ to be

$$\mathrm{clique}_n(\Gamma) := |\{n\text{-clique in } \Gamma\}|\,.$$

The geometric dimension of A_Γ is given by the size of the maximal clique in Γ:

$$\mathrm{gd}(A_\Gamma) = \dim_{\mathrm{CW}}(\mathrm{Sal}_\Gamma) = \max\{n \in \mathbb{N} \mid \mathrm{clique}_n(\Gamma) \neq 0\} \in \mathbb{N},$$

Here, "$\leq$" is witnessed by the Salvetti complex Sal_Γ, and "$\geq$" follows from the computation of the group homology below and Lemma 2.27. In particular, right-angled Artin groups are of type F.

Given a recipe to build a classifying space Sal_Γ by gluing in tori, one easily computes the group homology of right-angled Artin groups. Every cell in the Salvetti complex is a cycle, so that

$$H_n(\mathrm{Sal}_\Gamma;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{\mathrm{clique_n}(\Gamma)} & \text{if } 1 \leq n \leq \mathrm{gd}(A_\Gamma), \\ 0 & \text{if } n > \mathrm{gd}(A_\Gamma). \end{cases} \tag{21}$$

Example 3.49. Consider the right-angled Artin groups from Example 3.43. Notice that 1-cliques are vertices and 2-cliques are edges.

- Sal_{Γ_1} is a wedge of four circles $\bigvee^4 \mathbb{S}^1$, and the group homology coincides with the usual group homology of free groups (Example 2.15).
- Sal_{Γ_2} is a 4-torus $\mathbb{T}^4$, and the group homology coincides with the usual group homology of free Abelian groups (Example 2.14).
- Sal_{Γ_3} is a wedge of two 2-tori $\mathbb{T}^2 \vee \mathbb{T}^2$, and the group homology coincides with the usual group homology of free products of groups (Corollary 3.29).
- Sal_{Γ_4} is given by a wedge of four circles with three glued in tori according to the three commutator relations. The group homology of A_{Γ_4} is

$$H_n(A_{\Gamma_4}) = \begin{cases} \mathbb{Z}^4 & \text{if } \ n = 1, \\ \mathbb{Z}^3 & \text{if } \ n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

- Similarly, one computes for A_{Γ_5}

$$H_n(A_{\Gamma_5}) = \begin{cases} \mathbb{Z}^4 & \text{if } \ n = 1, \\ \mathbb{Z}^4 & \text{if } \ n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

- The group homology of A_{Γ_6} is given by

$$H_n(A_{\Gamma_6}) = \begin{cases} \mathbb{Z}^{12} & \text{if } \ n = 1, \\ \mathbb{Z}^8 & \text{if } \ n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

Example 3.50 (Betti numbers of products of free groups)**.** Let $m, m' \in \mathbb{N}_{\geq 1}$ and consider the group $G := F_m \times F_{m'}$. By the canonical presentation of G, we see that it is a right-angled Artin group with underlying graph Γ isomorphic to a complete bipartite graph $K_{m,m'}$. In particular, using the computation of group homology of right-angled Artin groups (21) we get that

$$\begin{aligned} \chi(A_\Gamma) &= b_2(A_\Gamma; \mathbb{Q}) - b_1(A_\Gamma; \mathbb{Q}) + b_0(A_\Gamma; \mathbb{Q}) \\ &= |\{\text{edges of } \Gamma\}| - |\{\text{vertices of } \Gamma\}| + 1 \\ &= m \cdot m' - (m + m') + 1. \end{aligned}$$

This closes a gap in the computation in Remark 2.44.

Remark 3.51 (group-homological properties of RAAGs)**.** Equation (21) shows that computing the group homology of a right-angled Artin group A_Γ amounts to the combinatorial problem of counting the number of n-cliques in the defining graph Γ. There is a natural way to turn simplicial graphs into higher-dimensional simplicial complexes called *flag complexes*, which we will introduce in Definition 3.54. Relating singular homology groups of these resulting flag complexes to the homology groups of the corresponding right-angled Artin group has been an important tool to produce groups with exotic properties. Most notably, the *Bestvina–Brady theorem* was used to exhibit examples of certain normal subgroups of right-angled Artin groups (so-called *Bestvina–Brady groups*)

- of type F_n, but not of type FP_{n+1}, and
- of type FP, but not of type F_2,

by reducing the problem to finding topological spaces with exotic homology groups [BB97]. We will see in Chapter 3.2.2, that very similar behavior turns out to hold for homology growth of right-angled Artin groups.

Finally, we observe that right-angled Artin groups are residually finite, and therefore admit the study of invariant growth within our framework.

Lemma 3.52 (residual finiteness of right-angled Artin groups)**.** *Right-angled Artin groups are residually finite.*

Sketch of proof. It can be shown that every right-angled Artin group A is *linear*, i.e., there exists $n \in \mathbb{N}$ and an injective homomorphism $A \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ [HW99, Corollary 3.6]. The claim then follows from the fact that $\mathrm{GL}_n(\mathbb{Z})$ is residually finite (Example 1.5) and that residual finiteness is preserved under passing to subgroups (Proposition 1.7). □

3.2.2. Main results

In the following, we state the three main results computing the rank gradient, Betti number growth and torsion growth of arbitrary right-angled Artin groups. The following result due to Kar–Nikolov holds more generally for Artin groups.

Theorem 3.53 (rank gradient of RAAGs [KN14, Lemma 1])**.** *Let Γ be a finite simplicial, let d be the number of connected components of Γ, and let A_Γ be the associated right-angled Artin group. Then for every $N_* \in \mathcal{R}(A_\Gamma)$ we have*

$$\mathrm{RG}(A_\Gamma, N_*) = d - 1.$$

Sketch of proof. It suffices to consider the case where $d = 1$. Let $\Gamma_1, \ldots, \Gamma_d$ be the connected components of Γ. Then $A_\Gamma \cong A_{\Gamma_1} * \ldots * A_{\Gamma_d}$ and by Corollary 1.61 we have for every residual chain that

$$\mathrm{RG}(A_\Gamma, N_*) = \mathrm{RG}(A_{\Gamma_1}, N_*^1) + \ldots + \mathrm{RG}(A_{\Gamma_d}, N_*^d) + (d-1) = d-1.$$

From now on assume that Γ is connected. We argue by induction over the number of vertices of Γ. If $|\Gamma| = 1$, then $A_\Gamma \cong \mathbb{Z}$, and in particular $\mathrm{RG}(A_\Gamma, N_*) = 0$ for every residual chain (Example 1.60). If $|\Gamma| = 2$, then A_Γ is of finite type. By Garside theory, its center is infinite cyclic [Cha16, p. 5]. This implies that $\mathrm{RG}(A_\Gamma, N_*) = 0$ [KN14, Proposition 1]. Let $|\Gamma| = n \geq 3$ and assume the claim holds for all graphs with at most $n-1$ vertices. Let $v \in \Gamma$ be a vertex. If removing v from Γ disconnects the graph, write $\Gamma_1, \ldots, \Gamma_r$ for the connected components of $\Gamma \setminus \{v\}$. Then

$$A_\Gamma \cong A_{\Gamma_1} *_{A_v} \ldots *_{A_v} A_{\Gamma_r}.$$

By induction hypothesis and the fact that the above is an amalgamated product over $A_v \cong \mathbb{Z}$, we get that $\mathrm{RG}(A_\Gamma, N_*) = 0$ [KN14, Proposition 2].

Now assume that removing v does not disconnect Γ. Let $(v, w) \in \Gamma$ be an edge. Then, A_e and $A_{\Gamma \setminus \{v\}}$ generate A_Γ. They intersect in $A_w \cong \mathbb{Z}$, and the claim follows from the induction hypothesis and a result of Abért–Jaikin–Nikolov [AJN11, Proposition 9]. $\square$

Definition 3.54 (flag complex, flag completion)**.** A *(finite) flag complex* L is a (finite) geometric simplicial complex with the property that every finite set of vertices that are pairwise connected by a 1-simplex, spans a simplex in L. There is a correspondence between finite simplicial graphs and finite flag complexes:

$$\begin{aligned} \{\text{finite simplicial graphs}\} &\xrightarrow{\sim} \{\text{finite flag complexes}\} \\ \Gamma &\mapsto \mathrm{flag}(\Gamma) \\ \text{1-skeleton } L^{(1)} &\leftarrow\!\shortmid L, \end{aligned}$$

where the *flag completion* $\mathrm{flag}(\Gamma)$ *of* Γ is given by the unique maximal flag complex with 1-skeleton Γ. In light of this correspondence, we will speak of the right-angled Artin group $A_L := A_{L^{(1)}}$ *associated to the flag complex* L.

Example 3.55 (flag complexes)**.**

- Consider the graphs in Figure 2. We have that
  - Γ_1, Γ_3, Γ_4 and Γ_5 are flag.
  - The flag completion of Γ_2 is a 2-simplex.
  - The flag completion of Γ_6 is a regular octahedron.

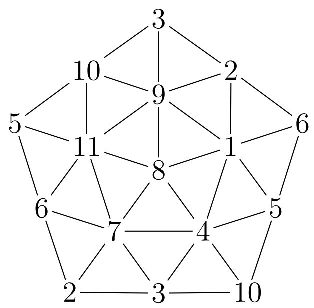


Figure 3: A minimal flag triangulation of $\mathbb{R}P^2$

We have seen that Γ_6 results from taking a "suspension-like" join of Γ_5 with a graph with two disjoint points. Iterating this procedure, we see that every sphere is *flag-triangulable*, i.e., homeomorphic to a finite flag complex.

- Every cycle-free simplicial graph is flag, because every full subgraph on more than 3 vertices contains a cycle and full subgraphs with one or two vertices impose no condition.
- We will see in Chapter 3.2.3, that many "nice" spaces are homotopy-equivalent to a flag complex.

Example 3.56 (flag triangulation of real projective space)**.** Consider the simplicial graph with 11 vertices and 45 edges (after the indicated identifications) in Figure 3. One can show that its flag completion is homeomorphic to the real projective space $\mathbb{R}P^2$ [Bib+20]. In particular, there exists a flag complex with torsion in its first homology group. More generally, Katzmann showed that [Ada14, Theorem 1]:

- If Γ is a simplicial graph with at most 10 vertices, then $H_1(\operatorname{flag}\Gamma;\mathbb{Z})$ is torsion-free.
- There exist exactly four simplicial graphs Γ (up to isomorphism) with 11 vertices for which $H_1(\operatorname{flag}\Gamma;\mathbb{Z})$ contains torsion.

Theorem 3.57 (Betti number growth of RAAGs [AOS21, Theorem 1])**.** *Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be a field. Let A_Γ be a right-angled Artin group associated to a flag complex $L = \operatorname{flag}\Gamma$. Then, for every residual chain $N_* \in \mathcal{R}(A_\Gamma)$ one has*

$$\widehat{b}_n\,(A_\Gamma, N_*;\mathbb{F}) = \widetilde{b}_{n-1}(L;\mathbb{F}).$$

In particular, $\widehat{b}_n\,(A_\Gamma;\mathbb{F})$ is a chain-independent proper limit.

Remark 3.58 (degree 1)**.** As a direct consequence of Theorem 3.57, every right-angled Artin group A_Γ has $\widehat{b_1}(A_\Gamma;\mathbb{Q}) = \widehat{b_1}(A_\Gamma;\mathbb{F}_p)$ for all primes p. In particular, Question 3.9 has a negative answer in the class of all right-angled Artin groups.

Example 3.59 (positive torsion growth, Avramidi–Okun–Schreve [AOS21])**.** Consider the finite simplicial graph Γ in Figure 3. Let $A = A_\Gamma$ be the associated right-angled Artin group. One can compute the singular homology groups via cellular homology as [Hat02, Examlpe 2.42]

$$H_n(\mathbb{R}P^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Applying Theorem 3.57 to $L = \operatorname{flag}(\Gamma) \cong_{\text{homeo}} \mathbb{R}P^2$, we compute

$$\widehat{b_3}(A;\mathbb{Q}) = \widetilde{b}_2(\mathbb{R}P^2;\mathbb{Q}) = \dim_{\mathbb{Q}} H_2(\mathbb{R}P^2;\mathbb{Q}) = 0,$$

and by using formula (14) we obtain

$$\widehat{b_3}(A;\mathbb{F}_2) = \widetilde{b}_2(\mathbb{R}P^2;\mathbb{F}_2) = \operatorname{rk}_{\mathbb{Z}} H_2(\mathbb{R}P^2;\mathbb{Z}) + \operatorname{mult}_2 H_2(\mathbb{R}P^2;\mathbb{Z}) + \operatorname{mult}_2 H_1(\mathbb{R}P^2;\mathbb{Z}) = 1.$$

Notice that $\operatorname{gd}(A) = 3$ by Example 3.48 and thus $\operatorname{cd}(A) \leq 3$. By Lemma 2.27, $H_3(A;\mathbb{Z})$ must be torsion-free. In particular, $\widehat{t_3}(A, N_*) = 0$. By Corollary 3.21 we get that:

$$\widehat{t_2}(A, N_*) \geq \left(\widehat{b_3}(A, N_*;\mathbb{Q}) - \widehat{b_3}(A, N_*;\mathbb{F}_2)\right) \cdot \log(2) = \log(2).$$

This example due to Avramidi, Okun and Schreve is likely the first known instance of a finitely presented group with positive torsion growth in degree ≥ 1, and for which the Betti number growth depends on the characteristic of the coefficient field. By Example 3.56, A is a right-angled Artin group with positive torsion growth in degree 2 with the minimal possible rank 11.

Theorem 3.60 (torsion growth of RAAGs)**.** *Let $n \in \mathbb{N}$. Let A_Γ be a right-angled Artin group associated to a flag complex $L = \operatorname{flag}(\Gamma)$. Then, for every residual chain $N_* \in \mathcal{R}(A_\Gamma)$ one has*

$$\widehat{t_n}(A_\Gamma, N_*) = \log \operatorname{tors} H_{n-1}(L;\mathbb{Z}).$$

In particular, $\widehat{t_n}(A_\Gamma)$ is a chain-independent proper limit.

Remark 3.61 (degree 1)**.** As a direct consequence of Corollary 3.60, every right-angled Artin group A_Γ has $\widehat{t_1}(A_\Gamma) = 0$. In the case where Γ is connected, this has been previously shown by Abert, Gelander and Nikolov [AGN17, Theorem 4]. In particular, Question 3.26 has a negative answer in the class of all right-angled Artin groups.

Example 3.62 (positive torsion growth)**.** By virtue of Corollary 3.60, Okun and Schreve have shown that the initial lower bound in the computation of Example 3.59 is indeed sharp: for A_Γ with Γ the graph from Figure 3 one has

$$\widehat{t_2}\,(A_\Gamma) = \log \operatorname{tors} H_1(\mathbb{R}P^2; \mathbb{Z}) = \log \operatorname{tors} \mathbb{Z}/2 = \log 2.$$

This is the likely the first example of a finitely presented group where the exact value of its positive torsion growth was computed.

3.2.3. Flag approximations

To make full use of the results of the previous section, our task is to come up with flag complexes with interesting homology groups. We will see that any reasonable prescribed sequence of Abelian groups is realized as the sequence of homology groups of a flag complex (Corollary 3.68). This can be shown by constructing such flag complexes explicitly. Instead, we start with the familiar class of CW-complexes, which can be transformed into flag complexes by first applying a simplicial approximation and then performing a barycentric subdivision.

Definition 3.63 (Moore space)**.** Let $n \in \mathbb{N}_{\geq 1}$ and let A be a $\mathbb{Z}$-module. We call a topological space X an $M(A, n)$-*Moore space*, if

- $H_j(X; \mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } \ j = 0, \\ A & \text{if } \ j = n, \\ 0 & \text{otherwise;} \end{cases}$
- and X is simply-connected if $n \geq 2$.

Theorem 3.64 (existence of Moore spaces [Hat02, Example 2.40])**.** *Let $n \in \mathbb{N}_{\geq 1}$ and let A be a $\mathbb{Z}$-module. Then, an $M(A, n)$-Moore space exists and can be constructed as a path-connected CW-complex, such that it is finite if A is finitely generated.*

Corollary 3.65 (realizability by CW-complexes [Hat02, Example 2.41])**.** *Given a sequence of $\mathbb{Z}$-modules $(A_i)_{i \in \mathbb{N}_{\geq 1}}$ there, exists a path-connected CW-complex X such that*

$$H_j(X; \mathbb{Z}) \cong_{\mathbb{Z}} A_j \textit{ for all } j \in \mathbb{N}_{\geq 1},$$

such that X has finite n-skeleton for $n \in \mathbb{N}_{\geq 1}$, if $A_1, \dots, A_n$ are finitely generated.

Sketch of proof. For every $i \geq 1$, let X_i be a path-connected CW-complex that is an $M(A_i, i)$-Moore space. Then forming the infinite wedge $X := \bigvee_{i \in \mathbb{N}_{\geq 1}} X_i$ is a space with the desired property; this follows from the fact that homology is additive under taking wedge products and the properties of Moore spaces. $\square$

Theorem 3.66 (simplicial approximability of CW-complexes [Hat02, Thm 2C.5])**.** *Every CW-complex X is homotopy equivalent to a simplicial complex K such that*

- $\dim_{\mathrm{SC}} K = \dim_{\mathrm{CW}} X$, *and*
- *K is finite if X is finite.*

Here, $\dim_{\mathrm{SC}} K$ *denotes the maximal dimension of a simplex of K, and* $\dim_{\mathrm{CW}} X$ *denotes the maximal dimension of a cell of X.*

Example 3.67 (barycentric subdivision)**.** A simplicial complex K is *flag* if every full subgraph of its 1-skeleton spans a simplex. If K fails this property, there is a full subgraph of the 1-skeleton that is not the 1-skeleton of any simplex. Subdividing simplices by inserting their barycentres (in particular, subdividing each edge by adding its midpoint — equivalently replacing an edge by a path on three vertices) does not change the homeomorphism type of K, but it removes such obstructions. The resulting *barycentric subdivision* is a flag complex homeomorphic to K. See Figure 4 and [Dav02, Example 3.7].

Corollary 3.68 (realizability by flag complexes)**.** *For any sequence of $\mathbb{Z}$-modules $(A_i)_{i\in\mathbb{N}_{\geq 1}}$ there exists a connected flag complex L such that*

$$H_j(L;\mathbb{Z}) \cong_{\mathbb{Z}} A_j \text{ for all } j \in \mathbb{N}_{\geq 1}.$$

If A_j is finitely generated for all $j \leq n$, then L has finite n-skeleton.

Proof. Start with a CW-complex X with the above property which exists by Corollary 3.65. By Theorem 3.66, there exists a simplicial complex K which is homotopy equivalent to K. By Example 3.67, its barycentric subdivision L is a flag complex homeomorphic to K. In particular, $H_j(L;\mathbb{Z}) \cong_{\mathbb{Z}} H_j(X;\mathbb{Z}) \cong_{\mathbb{Z}} A_j$ for all $j \in \mathbb{N}_{\geq 1}$. Moreover, these constructions can easily be seen to preserve finiteness of n-skeleta. □

Caveat 3.69. (flag-triangulability of CW-complexes) In general, only *regular* CW-complexes are *flag-triangulable* (i.e., are homeomorphic to a flag-complex) [FP90, Theorem 3.4.1]. There exists a CW-complex with five 0-cells, five 1-cells and one 2-cell that is not triangulable [FP90, p. 128].

3.2.4. Direct products

The formula relating torsion growth of a right-angled Artin group to the homology of the underlying flag complex allows us to describe inheritance results for homological invariant growth $\widehat{t_n}(-)$ and $\widehat{b_n}(-;\mathbb{F})$ under free products and direct products.

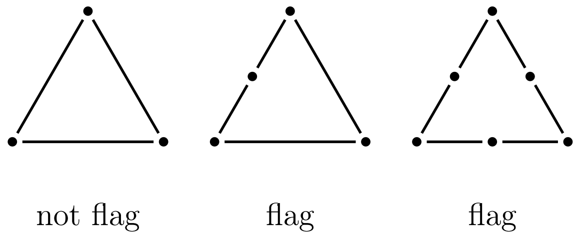


Figure 4: Barycentric subdivision and the flag condition.

For free products of right-angled Artin groups, this recovers our more general results from Chapter 3.1.5, and gives it a nice geometric interpretation. We now hope to gain some intuition as to what behavior one might expect from direct products of groups, assuming that the case of right-angled Artin groups is again representative of the general case. We have seen that taking a direct product of right-angled Artin groups corresponds to forming the join of the underlying graphs. To make use of the main results, we have to show that this is well-behaved with respect to flag completions.

Definition 3.70 (topological join)**.** Let X and Y be non-empty topological spaces. The *join* $X * Y$ *of* X *and* Y is given by

$$(X \times Y \times [0,1])/\sim,$$

where $\sim$ is an equivalence relation generated by

$$\begin{aligned}(x,y,0) &\sim (x,y',0) \text{ for all } x \in X \text{ and } y,y' \in Y,\\ (x,y,1) &\sim (x',y,1) \text{ for all } x,x' \in X \text{ and } y \in Y.\end{aligned}$$

This construction can be thought of "attaching line segments joining every point in X with every point in Y".

Remark 3.71 (topological/simplicial/graph join)**.** Given two geometric simplicial complexes X and Y, their join $X * Y$ can be naturally endowed with the structure of a geometric simplicial complex itself. Moreover, there is a join operation on abstract simplicial complexes, which is compatible with the geometric realization functor. For example, given two simplicial graphs Γ and Λ, one defines their (combinatorial) join $\Gamma * \Lambda$ with vertices $V(\Gamma * \Lambda) := V(\Gamma) \sqcup V(\Lambda)$, and edges

$$E(\Gamma * \Lambda) := E(\Gamma) \sqcup E(\Lambda) \sqcup \big\{\{\gamma,\lambda\} \mid \gamma \in V(\Gamma), \lambda \in V(\Lambda)\big\}.$$

This is simply the graph consisting of a copy of Γ and a copy of Λ, with all vertices between those copies connected by edges. Then the geometric realization of $\Gamma * \Lambda$ is isomorphic (as a geometric simplicial complex) to the topological join of the geometric realizations of Γ and Λ equipped with the natural structure of a geometric simplicial complex.

Example 3.72 (suspension is a join)**.** A special case of a join is the join with the graph consisting of two non-adjacent vertices $\mathbb{S}^0$. For a simplicial complex X, we have that $X * \mathbb{S}^0 \cong \Sigma X$.

Proposition 3.73 (join of flag complexes [Dav12, Lemma A.4.7])**.** *Given two finite simplicial graphs Γ and Λ, we have that*

$$\operatorname{flag}(\Gamma \sqcup \Lambda) \cong \operatorname{flag}(\Gamma) \sqcup \operatorname{flag}(\Lambda) \text{ and } \operatorname{flag}(\Gamma * \Lambda) \cong \operatorname{flag}(\Gamma) * \operatorname{flag}(\Lambda).$$

In particular, the 1-skeletons are isomorphic.

This recovers the fact that torsion growth is additive under free products (Theorem 3.30) in the case of right-angled Artin groups from a more topological viewpoint.

Corollary 3.74 (torsion growth of free products of RAAGs)**.** *Let $n \in \mathbb{N}$ and let A_Γ and A_Λ be right-angled Artin groups associated to finite simplicial graphs Γ and Λ, respectively. Then*

$$\widehat{t_n}\left(A_\Gamma * A_\Lambda\right) = \widehat{t_n}\left(A_\Gamma\right) + \widehat{t_n}\left(A_\Lambda\right).$$

Proof. For $n = 0$ the claim is clear. For $n \in \mathbb{N}_{\geq 1}$, this follows from combining our previous observations:

$$\begin{aligned}
\widehat{t_n}\left(A_\Gamma * A_\Lambda\right) &= \widehat{t_n}\left(A_{\Gamma \sqcup \Lambda}\right) && \text{(Lemma 3.44)}\\
&= \log \operatorname{tors} H_{n-1}\big(\operatorname{flag}(\Gamma \sqcup \Lambda)\big) && \text{(Corollary 3.60)}\\
&= \log \operatorname{tors} H_{n-1}\big(\operatorname{flag}\Gamma \sqcup \operatorname{flag}\Lambda\big) && \text{(Proposition 3.73)}\\
&= \log \operatorname{tors} \big(H_{n-1}(\operatorname{flag}\Gamma) \oplus H_{n-1}(\operatorname{flag}\Lambda)\big) && \text{(homology and } \sqcup)\\
&= \log \operatorname{tors} H_{n-1}(\operatorname{flag}\Gamma) + \log \operatorname{tors} H_{n-1}(\operatorname{flag}\Lambda) && (\log \operatorname{tors} \text{ and } \oplus)\\
&= \widehat{t_n}\left(A_\Gamma\right) + \widehat{t_n}\left(A_\Lambda\right). && \text{(Corollary 3.60)}
\end{aligned}$$

□

The same follows for Betti number growth. Notice that non-trivial right-angled Artin groups are infinite groups, so that the extra terms in the first Betti number growth of free products vanish. Now let us consider the case of direct products of right-angled Artin groups. The topological join satisfies a Künneth-like formula.

Theorem 3.75 (Künneth theorem for join [Enc]). *Let R be a principal ideal doain. Let X and Y be topological spaces. Then for $n \geq 1$ we have:*

$$\begin{aligned} \widetilde{H}_n(X * Y; R) \cong_R & \bigoplus_{i+j=n-1} \widetilde{H}_i(X; R) \otimes_R \widetilde{H}_j(Y; R) \\ & \oplus \bigoplus_{i+j=n-2} \operatorname{Tor}_1^R\big(\widetilde{H}_i(X; R), \widetilde{H}_j(Y; R)\big). \end{aligned} \tag{22}$$

Corollary 3.76. *Let $n \in \mathbb{N}_{\geq 1}$, let $\mathbb{F}$ be a field, and let A_Γ and A_Λ be two right-angled Artin groups.*

1. *If $A_\Gamma, A_\Lambda \in \mathbf{H}_{n-1}(\mathbb{F})$, then $A_\Gamma \times A_\Lambda \in \mathbf{H}_n(\mathbb{F})$.*
2. *If $A_\Gamma, A_\Lambda \in \mathbf{T}_{n-1}$, then $A_\Gamma \times A_\Lambda \in \mathbf{T}_n$.*

Proof. Ad 1. By assumption, we have that for all $m \in \{0, \dots, n-1\}$

$$\widetilde{b}_{m-1}(\operatorname{flag}\Gamma; \mathbb{F}) = \widehat{b}_m\,(A_\Gamma; \mathbb{F}) = 0,$$

and similarly for Λ. We compute for all $m \in \{0, \dots, n\}$

$$\begin{aligned} \widehat{b}_m\,(A_\Gamma \times A_\Lambda; \mathbb{F}) &= \widehat{b}_m\,(A_{\Gamma * \Lambda}; \mathbb{F}) && \text{(Lemma 3.45)} \\ &= \widetilde{b}_{m-1}\big(\operatorname{flag}(\Gamma * \Lambda); \mathbb{F}\big) && \text{(Theorem 3.57)} \\ &= \widetilde{b}_{m-1}\big(\operatorname{flag}\Gamma * \operatorname{flag}\Lambda; \mathbb{F}\big) && \text{(Proposition 3.73)} \\ &= \dim_{\mathbb{F}} \widetilde{H}_{m-1}\big(\operatorname{flag}\Gamma * \operatorname{flag}\Lambda; \mathbb{F}\big) && \text{(Definition 2.41)} \\ &= \sum_{i+j=m-2} \dim_{\mathbb{F}} \widetilde{H}_i(\operatorname{flag}\Gamma; \mathbb{F}) \cdot \dim_{\mathbb{F}} \widetilde{H}_j(\operatorname{flag}\Lambda; \mathbb{F}) && \text{(Lemma 2.40)} \\ &= \sum_{i+j=m} \widetilde{b}_{i-1}(\operatorname{flag}\Gamma; \mathbb{F}) \cdot \widetilde{b}_{j-1}(\operatorname{flag}\Lambda; \mathbb{F}) && \text{(re-indexing)} \\ &= 0. && \text{(assumption)} \end{aligned}$$

Ad 2. By assumption, we have that for all $m \in \{0, \dots, n-1\}$

$$\log \operatorname{tors} H_{m-1}(\operatorname{flag}\Gamma; \mathbb{Z}) = \widehat{t}_m\,(A_\Gamma) = 0,$$

and similarly for Λ. In particular, $\operatorname{flag}\Gamma$ and $\operatorname{flag}\Lambda$ have free (reduced) homology groups up to degree $m-1$. As tensor products of free modules are free, and $\operatorname{Tor}_1^{\mathbb{Z}}$ vanishes on free modules, the Künneth formula (22) implies that

$$\widetilde{H}_{m-1}(\operatorname{flag}\Gamma * \operatorname{flag}\Lambda)$$

is free for all $m \in \{0, \dots, n\}$. The claim follows easily:

$$\begin{aligned}\widehat{t}_m\,(A_\Gamma \times A_\Lambda) &= \widehat{t}_m\,(A_{\Gamma * \Lambda}) && \text{(Lemma 3.45)}\\ &= \log \operatorname{tors} H_{m-1}\big(\operatorname{flag}(\Gamma * \Lambda)\big) && \text{(Theorem 3.57)}\\ &= \log \operatorname{tors} H_{m-1}\big(\operatorname{flag}\Gamma * \operatorname{flag}\Lambda\big) && \text{(Proposition 3.73)}\\ &= 0. \qquad \square\end{aligned}$$

We will see in the next chapter, that analogous statements hold for all residually finite groups, under suitable assumptions on finiteness type and chain-independence.

3.3. Homology growth of direct products

In this chapter, we compute homology growth of direct products of groups. We review the *Künneth theorem*, which relates the group homology of direct products to the homology groups of the factors: it consists of tensor product contributions and possible Tor-terms. In the case of Betti numbers, the contributions of the pure torsion modules $\operatorname{Tor}_1^{\mathbb{Z}}(-)$ vanish, which leads to a Künneth-type formula for Betti number growth of direct products (Theorem 3.82). In the torsion case, the situation is more delicate: not only do the $\operatorname{Tor}_1^{\mathbb{Z}}(-)$ modules contribute non-trivially, but this contribution depends on the actual canonical decompositions of the homology modules, not only on the cardinalities of the torsion submodules. For this reason, we only obtain upper and lower bounds – but these still lead to interesting results in special cases. (Theorems 3.88 and 3.90).

3.3.1. Künneth theorem

Theorem 3.77 (Künneth theorem for group homology)**.** *Let G and H be groups, let R be a principal ideal domain (with trivial G- and H-action), and let $n \in \mathbb{N}$. Then (where the R-module structure is inherited from the coefficients)*

$$H_n(G\times H;R) \cong_R \bigoplus_{i+j=n} H_i(G;R)\otimes_R H_j(H;R) \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}_1^R\big(H_i(G;R), H_j(H;R)\big). \tag{23}$$

If $R = \mathbb{F}$ is a field, then:

$$H_n(G \times H;\mathbb{F}) \cong_{\mathbb{F}} \bigoplus_{i+j=n} H_i(G;\mathbb{F}) \otimes_{\mathbb{F}} H_j(H;\mathbb{F}). \tag{24}$$

Proof. The proof is quite involved and uses spectral sequence arguments. Details can be found in Rotman's book [Rot09, Chapter 10.10]. (24) follows easily from (23) by using the fact that $\operatorname{Tor}_1^{\mathbb{F}}$ vanishes for free $\mathbb{F}$-vector spaces (Proposition 2.46). $\square$

Our goal now is to understand how to compute the homological invariants of $H_n(G \times H; R)$ from equation (23). For this we need to understand how $\otimes_R$ and Tor_1^R affect the homological invariants.

Lemma 3.78 (torsion and tensor products)**.** *Let A and B be finitely generated $\mathbb{Z}$-modules with canoncial decompositions*

$$A \cong_{\mathbb{Z}} \mathbb{Z}^r \oplus \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i} \quad \text{and} \quad B \cong_{\mathbb{Z}} \mathbb{Z}^s \oplus \bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}.$$

Then the $\mathbb{Z}$-module $A \otimes_{\mathbb{Z}} B$ decomposes as:

$$A \otimes_{\mathbb{Z}} B \cong_{\mathbb{Z}} \mathbb{Z}^{rs} \oplus \Big(\bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big)^s \oplus \Big(\bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big)^r \oplus \bigoplus_{\substack{(i,j) \in \mathcal{I} \times \mathcal{J}: \\ p_i = q_j}} \mathbb{Z}/p_i^{\min\{n_i, m_j\}}.$$

In particular, we have that

$$\mathrm{tors}(A \otimes_{\mathbb{Z}} B) = (\mathrm{tors}\, A)^s \cdot (\mathrm{tors}\, B)^r \cdot \prod_{\substack{(i,j) \in \mathcal{I} \times \mathcal{J}: \\ p_i = q_j}} p_i^{\min\{n_i, m_j\}}$$

and

$$\mathrm{rk}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B) = \mathrm{rk}_{\mathbb{Z}}(A) \cdot \mathrm{rk}_{\mathbb{Z}}(B).$$

Proof. We have the following

$$\begin{aligned}
A \otimes_{\mathbb{Z}} B &\cong_{\mathbb{Z}} \Big(\mathbb{Z}^r \oplus \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big) \otimes_{\mathbb{Z}} \Big(\mathbb{Z}^s \oplus \bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big) \\
&\cong_{\mathbb{Z}} (\mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{Z}^s) \oplus \Big(\bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big) \otimes_{\mathbb{Z}} \mathbb{Z}^s \oplus \mathbb{Z}^r \otimes_{\mathbb{Z}} \Big(\bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big) \\
&\quad \oplus \Big(\bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big) \otimes_{\mathbb{Z}} \Big(\bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big) \\
&\cong_{\mathbb{Z}} \mathbb{Z}^{rs} \oplus \Big(\bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big)^s \oplus \Big(\bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big)^r \oplus \bigoplus_{(i,j) \in \mathcal{I} \times \mathcal{J}} (\mathbb{Z}/p_i^{n_i} \otimes_{\mathbb{Z}} \mathbb{Z}/q_j^{m_j}) \\
&\cong_{\mathbb{Z}} \mathbb{Z}^{rs} \oplus \Big(\bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i}\Big)^s \oplus \Big(\bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big)^r \oplus \bigoplus_{(i,j) \in \mathcal{I} \times \mathcal{J}} \mathbb{Z}/\gcd(p_i^{n_i}, q_j^{m_j}),
\end{aligned}$$

where we have used well-known properties of the tensor product. Since the p_i and q_j are primes, we have that

$$\gcd\left(p_i^{n_i}, q_j^{m_j}\right) = \begin{cases} p_i^{\min\{n_i, m_j\}} & \text{if } p_i = q_j, \\ 1 & \text{otherwise;} \end{cases}$$

and the claimed decomposition follows. The statement about the cardinality of torsion modules follows from Proposition 2.35, and the statement about ranks follows from the uniqueness of the decomposition of finitely generated $\mathbb{Z}$-modules. □

Caveat 3.79. In general, tors of a tensor product is not uniquely determined by tors of the factors. For example: let $A = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $B = \mathbb{Z}/4$. Then we have $\operatorname{tors} A = \operatorname{tors} B = 4$, but $\operatorname{tors}(A \otimes_{\mathbb{Z}} A) = 16$ and $\operatorname{tors}(B \otimes_{\mathbb{Z}} B) = 4$.

Corollary 3.80. *Let A and B be finitely generated $\mathbb{Z}$-modules with $s = \operatorname{rk}_{\mathbb{Z}} B$ and $r = \operatorname{rk}_{\mathbb{Z}} A$. Then*

$$(\operatorname{tors} A)^s \cdot (\operatorname{tors} B)^r \leq \operatorname{tors}(A \otimes_{\mathbb{Z}} B) \leq (\operatorname{tors} A)^s \cdot (\operatorname{tors} B)^r \cdot \exp\left(\frac{\log \operatorname{tors} A \cdot \log \operatorname{tors} B}{\log 2}\right).$$

Proof. We have seen above that given the canonical decompositions

$$A \cong_{\mathbb{Z}} \mathbb{Z}^r \oplus \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p_i^{n_i} \quad \text{and} \quad B \cong_{\mathbb{Z}} \mathbb{Z}^s \oplus \bigoplus_{j \in \mathcal{J}} \mathbb{Z}/q_j^{m_j},$$

we have that

$$\operatorname{tors}(A \otimes_{\mathbb{Z}} B) = (\operatorname{tors} A)^s \cdot (\operatorname{tors} B)^r \cdot \underbrace{\prod_{\substack{(i,j) \in \mathcal{I} \times \mathcal{J}: \\ p_i = q_j}} p_i^{\min\{n_i, m_j\}}}_{=: E(A,B)}.$$

Thus it suffices to show that

$$1 \leq E(A,B) \leq \exp\left(\frac{\log \operatorname{tors} A \cdot \log \operatorname{tors} B}{\log 2}\right).$$

The lower bound is clear, it is achieved if and only if the sets of primes $\{p_i \mid i \in \mathcal{I}\}$ and $\{q_j \mid j \in \mathcal{J}\}$ are disjoint.

To bound $E(A,B)$ above, write for every prime p

$$n_p = \sum_{i \in \mathcal{I}:\, p_i = p} n_i \quad \text{and} \quad m_p = \sum_{j \in \mathcal{J}:\, q_j = p} m_j,$$

so that

$$E(A,B) = \prod_p p^{\min\{n_p, m_p\}} \leq \prod_p \left(p^{n_p}\right)^{m_p} \leq \Big(\prod_p p^{n_p}\Big)^{\sum_{j \in \mathcal{J}} m_j} = (\operatorname{tors} A)^{\sum_{j \in \mathcal{J}} m_j},$$

where each product runs over all prime numbers p. Since each $q_j \geq 2$, $\operatorname{tors} B = \prod_{j\in\mathcal{J}} q_j^{m_j} \geq 2^{\sum_{j\in\mathcal{J}} m_j}$ and hence $\sum_{j\in\mathcal{J}} m_j \leq \log_2(\operatorname{tors} B)$. It follows that

$$\begin{aligned} E(A,B) &\leq (\operatorname{tors} A)^{\log_2(\operatorname{tors} B)} \\ &= \exp\big(\log(\operatorname{tors} A)\cdot \log_2(\operatorname{tors} B)\big) \\ &= \exp\left(\frac{\log \operatorname{tors} A \cdot \log \operatorname{tors} B}{\log 2}\right). \end{aligned}$$

□

Corollary 3.81 ([Rot09, Example 7.7.]). *Let A and B be finitely generated $\mathbb{Z}$-modules with canonical decompositions*

$$A \cong_{\mathbb{Z}} \mathbb{Z}^r \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i} \quad \text{and} \quad B \cong_{\mathbb{Z}} \mathbb{Z}^s \oplus \bigoplus_{j\in\mathcal{J}} \mathbb{Z}/q_j^{m_j}.$$

Then we have

$$\operatorname{Tor}_1^{\mathbb{Z}}(A,B) \cong_{\mathbb{Z}} \bigoplus_{\substack{(i,j)\in\mathcal{I}\times\mathcal{J}:\\ p_i=q_j}} \mathbb{Z}/p_i^{\min\{n_i,m_j\}}.$$

In particular,

$$1 \leq \operatorname{tors}\operatorname{Tor}_1^{\mathbb{Z}}(A,B) \leq \exp\left(\frac{\log \operatorname{tors} A \cdot \log \operatorname{tors} B}{\log 2}\right).$$

Proof. We compute

$$\begin{aligned} \operatorname{Tor}_1^{\mathbb{Z}}(A,B) &\cong_{\mathbb{Z}} \operatorname{Tor}_1^{\mathbb{Z}}\Big(\mathbb{Z}^r \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i}, \mathbb{Z}^s \oplus \bigoplus_{j\in\mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big) \\ &\cong_{\mathbb{Z}} \operatorname{Tor}_1^{\mathbb{Z}}\Big(\bigoplus_{i\in\mathcal{I}} \mathbb{Z}/p_i^{n_i}, \bigoplus_{j\in\mathcal{J}} \mathbb{Z}/q_j^{m_j}\Big) \\ &\cong_{\mathbb{Z}} \bigoplus_{(i,j)\in\mathcal{I}\times\mathcal{J}} \operatorname{Tor}_1^{\mathbb{Z}}\big(\mathbb{Z}/p_i^{n_i}, \mathbb{Z}/q_j^{m_j}\big) \\ &\cong_{\mathbb{Z}} \bigoplus_{\substack{(i,j)\in\mathcal{I}\times\mathcal{J}:\\ p_i=q_j}} \mathbb{Z}/p_i^{\min\{n_i,m_j\}}. \end{aligned}$$

Notice that $\operatorname{tors}\operatorname{Tor}_1^{\mathbb{Z}}(A,B) = E(A,B)$, and the claimed inequality follows from the proof Corollary 3.80. □

3.3.2. Betti number growth of direct products

Theorem 3.82 (Betti number growth of direct products)**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G_1 and G_2 be residually finite groups of type* FP_n*. Let $N^1_* \in \mathcal{R}(G_1)$ and $N^2_* \in \mathcal{R}(G_2)$ be residual chains. Then*

$$\widehat{b}_n\left(G_1 \times G_2, N^1_* \times N^2_*; \mathbb{F}\right) \leq \sum_{i+j=n} \widehat{b}_i\left(G_1, N^1_*; \mathbb{F}\right) \cdot \widehat{b}_j\left(G_2, N^2_*; \mathbb{F}\right).$$

If $\widehat{b}_i\left(G_1, N^1_*; \mathbb{F}\right)$ and $\widehat{b}_i\left(G_2, N^2_*; \mathbb{F}\right)$ are proper limits for all $i \in \{1, \dots, n\}$, then the same holds for $\widehat{b}_n\left(G_1 \times G_2, N^1_* \times N^2_*; \mathbb{F}\right)$ and the above becomes an equality.

Proof. By Lemma 1.15, $N^1_* \times N^2_* \in \mathcal{R}(G_1 \times G_2)$ is a residual chain. Using the Künneth formula (Theorem 3.77), the index formula for product chains (Lemma 1.15), and the standard properties of dimensions (Lemma 2.40) we obtain:

$$\begin{aligned}
\widehat{b}_n\left(G_1 \times G_2, N^1_k \times N^2_k; \mathbb{F}\right) &= \limsup_{k\to\infty} \frac{\dim_{\mathbb{F}} H_n(N^1_k \times N^2_k; \mathbb{F})}{[G_1 \times G_2 : N^1_k \times N^2_k]} \\
&= \limsup_{k\to\infty} \frac{\dim_{\mathbb{F}} \left(\bigoplus_{i+j=n} H_i(N^1_k; \mathbb{F}) \otimes_{\mathbb{F}} H_j(N^2_k; \mathbb{F})\right)}{[G_1 \times G_2 : N^1_k \times N^2_k]} \\
&= \limsup_{k\to\infty} \frac{\sum_{i+j=n} \dim_{\mathbb{F}} H_i(N^1_k; \mathbb{F}) \cdot \dim_{\mathbb{F}} H_j(N^2_k; \mathbb{F})}{[G_1 : N^1_k] \cdot [G_2 : N^2_k]} \\
&= \limsup_{k\to\infty} \left(\sum_{i+j=n} \frac{b_i(N^1_k; \mathbb{F})}{[G_1 : N^1_k]} \cdot \frac{b_j(N^2_k; \mathbb{F})}{[G_2 : N^2_k]}\right) \\
&\leq \sum_{i+j=n} \limsup_{k\to\infty} \left(\frac{b_i(N^1_k; \mathbb{F})}{[G_1 : N^1_k]} \cdot \frac{b_j(N^2_k; \mathbb{F})}{[G_2 : N^2_k]}\right) \\
&\leq \sum_{i+j=n} \widehat{b}_i\left(G_1, N^1_k; \mathbb{F}\right) \cdot \widehat{b}_j\left(G_2, N^2_k; \mathbb{F}\right).
\end{aligned}$$

In the penultimate step, we use that the involved terms are all non-negative (Proposition 1.28.2). In the last step, we use the fact that G_1 and G_2 are of type FP_n, and therefore both have bounded Betti number growths up to degree n by Corollary 3.22. In particular, the limits superior of the factors are not of the form $0 \cdot \infty$ (Proposition 1.28.3). If the involved terms are all proper limits, then we get equalities by the second and third parts of Proposition 1.28. □

Remark 3.83. One can weaken the assumption that G_1 and G_2 are of type FP_n in Corollary 3.82. We technically only need to ensure that there do not exist $i, j \in \{0, \dots, n\}$ with $i + j \leq n$ such that $\widehat{b}_i\left(G_1, N^1_*; \mathbb{F}\right) = 0$ and $\widehat{b}_j\left(G_2, N^2_*; \mathbb{F}\right) = +\infty$. This can be achieved naturally by assuming the finiteness types, as we then force all values to be finite. In specific applications or low degrees, where one has more information about the classes of groups involved, this might be replaced by weaker assumptions (e.g. in Corollary 3.85).

The above formula recovers a well-known result about L^2-Betti numbers of direct products of groups [Kam19, Theorem 4.15(i)].

Corollary 3.84 (L^2-Betti numbers of direct products)**.** *Let $n \in \mathbb{N}$ and let G_1 and G_2 be residually finite groups of type* FP_{n+1}*. Then*

$$b_n^{(2)}(G_1 \times G_2) = \sum_{i+j=n} b_i^{(2)}(G_1) \cdot b_j^{(2)}(G_2).$$

Proof. This follows immediately from Theorem 3.82 and Lück's approximation theorem 3.34. □

Corollary 3.85 (vanishing Betti number growth of direct products)**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field, and let G_1 and G_2 be residually finite groups. Suppose that*

1. *$G_1 \in \mathbf{H}_n(\mathbb{F})$ and G_2 is of type FP_n; or*
2. *G_1 and G_2 are of type FP_n, and there exists an $n_0 \in \{0, \ldots, n\}$ such that $G_1 \in \mathbf{H}_{n_0}(\mathbb{F})$ and $G_2 \in \mathbf{H}_{n-n_0-1}(\mathbb{F})$.*

If $\widehat{b}_m\left(G_1 \times G_2; \mathbb{F}\right)$ is chain-independent for all $m \in \{1, \ldots, n\}$, then $G_1 \times G_2 \in \mathbf{H}_n(\mathbb{F})$.

Proof. Consider the inequality in Theorem 3.82 for $m \in \{0, \ldots, n\}$

$$\widehat{b}_m\left(G_1 \times G_2, N_*^1 \times N_*^2; \mathbb{F}\right) \leq \sum_{i+j=m} \widehat{b}_i\left(G_1, N_*^1; \mathbb{F}\right) \cdot \widehat{b}_j\left(G_2, N_*^2; \mathbb{F}\right).$$

Ad 1. If $G_1 \in \mathbf{H}_n(\mathbb{F})$ and G_2 is of type FP_n, then every $\widehat{b}_i$-summand on the right side vanishes, and every $\widehat{b}_j$-summand is finite for all $m \in \{0, \ldots, n\}$. In particular, Theorem 3.82 applies (see Remark 3.83) and the right side vanishes.

Ad 2. As G_1 and G_2 are of type FP_n, we have that $\widehat{b}_i\left(G_l, N_*^l; \mathbb{F}\right) < \infty$ for all $i \in \{0, \ldots, n\}$. By assumption, for every $i \in \{0, \ldots, m\}$, we have that $G_1 \in \mathbf{H}_i(\mathbb{F})$ or $G_2 \in \mathbf{H}_{n-i}(\mathbb{F})$. Hence the upper bound is zero and the claim follows. □

Remark 3.86. The statement of Corollary 3.85.1 coincides with a special case of a bootstrapping result of Li–Löh–Moraschini–Sauer–Uschold [LLMSU24, Corollary 3.7.(ii)]. Their result is phrased more generally for group extensions. However, Corollary 3.85.2 appears to have no counterpart in their work.

3.3.3. Torsion growth of direct products

Corollary 3.87. *Let $n \in \mathbb{N}$ and let G and H be groups of type FP_{n+1}. Then $\log \operatorname{tors} H_n(G \times H)$ is bounded below by*

$$\sum_{i+j=n} \Big(b_j(H) \cdot \log \operatorname{tors} H_i(G) + b_i(G) \cdot \log \operatorname{tors} H_j(H) \Big);$$

and bounded above by

$$\begin{aligned} &\sum_{i+j=n} \Big(b_j(H) \cdot \log \operatorname{tors} H_i(G) + b_i(G) \cdot \log \operatorname{tors} H_j(H) \\ &+ (\log 2)^{-1} \cdot \log \operatorname{tors} H_i(G) \cdot \log \operatorname{tors} H_j(H) \Big) \\ &+ (\log 2)^{-1} \cdot \sum_{i+j=n-1} \log \operatorname{tors} H_i(G) \cdot \log \operatorname{tors} H_j(H). \end{aligned}$$

In particular: for $n = 1$ we have

$$\log \operatorname{tors} H_1(G \times H) = \log \operatorname{tors} H_1(G) + \log \operatorname{tors} H_1(H).$$

Proof. This follows directly from combining Theorem 3.77, Theorem 3.78 and Corollary 3.81, where we have rewritten $\operatorname{rk}_{\mathbb{Z}} H_*(-)$ as $b_*(-)$. □

Theorem 3.88 (lower bound for torsion growth of direct products)**.** *Let $n \in \mathbb{N}$, let G_1 and G_2 be residually finite groups of type FP_{n+1}, and let $N^1_* \in \mathcal{R}(G_1)$ and $N^2_* \in \mathcal{R}(G_2)$. Assume that $\widehat{t_i}\,(G_1, N^1_*)$ and $\widehat{t_i}\,(G_2, N^2_*)$ are proper limits for all $i \in \{1, \dots, n\}$. Then*

$$\widehat{t_n}\left(G_1 \times G_2, N^1_* \times N^2_*\right) \geq \sum_{i+j=n} \Big(b^{(2)}_j(G_2) \cdot \widehat{t_i}\left(G_1, N^1_*\right) + b^{(2)}_i(G_1) \cdot \widehat{t_j}\left(G_2, N^2_*\right) \Big).$$

Proof. Recall that $N^1_* \times N^2_* \in \mathcal{R}(G_1 \times G_2)$ by Lemma 1.15. For every $k \in \mathbb{N}$, applying Corollary 3.87 to $N^1_* \times N^2_*$ and dividing by

$$[G_1 \times G_2 : N^1_k \times N^2_k] = [G_1 : N^1_k] \cdot [G_2 : N^2_k] \neq 0$$

yields the inequality

$$\frac{\log \operatorname{tors} H_n(N^1_k \times N^2_k)}{[G_1 \times G_2 : N^1_k \times N^2_k]} \geq \sum_{i+j=n} \left(\frac{b_j(N^2_k)}{[G_2 : N^2_k]} \cdot \frac{\log \operatorname{tors} H_i(N^1_k)}{[G_1 : N^1_k]} + \frac{b_i(N^1_k)}{[G_1 : N^1_k]} \cdot \frac{\log \operatorname{tors} H_j(N^2_k)}{[G_2 : N^2_k]} \right).$$

The claim follows from passing to the lim sup, using that by Corollary 3.22 and Lück's approximation theorem 3.34, all involved terms converge to finite values, and that the $\mathbb{Q}$-Betti number growths coincide with the L^2-Betti numbers. □

Corollary 3.89. *Let $n \in \mathbb{N}$. Let G_1 and G_2 be residually finite groups of type* FP_n*. Assume that there exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 \leq n$ and $N^1_* \in \mathcal{R}(G_1)$ such that $\widehat{t}_{n_1}(G_1, N^1_*) > 0$ and $b^{(2)}_{n_2}(G_2) > 0$. Let $N^2_* \in \mathcal{R}(G_2)$ arbitrary. Then*

$$\widehat{t}_{n_1+n_2}\left(G_1 \times G_2, N^1_* \times N^2_*\right) > 0.$$

Proof. Notice that we *do not* need to assume that $\widehat{t}_{n_1}(G_1, N^1_*)$ is a proper limit: in the proof of Theorem 3.87, we have that

$$\frac{\log \operatorname{tors} H_{n_1+n_2}(N^1_k \times N^2_k)}{[G_1 \times G_2 : N^1_k \times N^2_k]} \geq \frac{b_{n_2}(G_2; \mathbb{Q})}{[G_2 : N^2_k]} \cdot \frac{\log \operatorname{tors} H_{n_1}(N^1_k)}{[G_1 : N^1_k]}.$$

Passing to the limit superior we obtain

$$\widehat{t}_{n_1+n_2}\left(G_1 \times G_2, N^1_* \times N^2_*\right) \geq b^{(2)}_{n_2}(G_2) \cdot \widehat{t}_{n_1}\left(G_1, N^1_*\right) > 0. \qquad \square$$

Theorem 3.90 (upper bound for torsion growth of direct product)**.** *Let $n \in \mathbb{N}$, and let G_1 and G_2 be residually finite groups of type* FP_{n+1}*. Let $N^1_* \in \mathcal{R}(G_1)$ and $N^2_* \in \mathcal{R}(G_2)$. Then*

$$\begin{aligned}\widehat{t}_n\left(G_1 \times G_2, N^1_* \times N^2_*\right) \leq \sum_{i+j=n} \Big(& b^{(2)}_j(G_2) \cdot \widehat{t}_i\left(G_1, N^1_*\right) + b^{(2)}_i(G_1) \cdot \widehat{t}_j\left(G_2, N^2_*\right) \\ & + (\log 2)^{-1} \cdot \widehat{t}_i\left(G_1, N^1_*\right) \cdot \widehat{t}_j\left(G_2, N^2_*\right) \Big) \\ & + (\log 2)^{-1} \cdot \sum_{i+j=n-1} \widehat{t}_i\left(G_1, N^1_*\right) \cdot \widehat{t}_j\left(G_2, N^2_*\right).\end{aligned}$$

Proof. Recall that $N^1_* \times N^2_* \in \mathcal{R}(G_1 \times G_2)$ by Lemma 1.15. For every $k \in \mathbb{N}$, applying Corollary 3.87 to $N^1_* \times N^2_*$ and dividing by $[G_1 \times G_2 : N^1_k \times N^2_k] = [G_1 : N^1_k] \cdot [G_2 : N^2_k] \neq 0$ gives

$$\begin{aligned}\frac{\log \operatorname{tors} H_n(N^1_k \times N^2_k)}{[G_1 \times G_2 : N^1_k \times N^2_k]} \leq \sum_{i+j=n} \Bigg(& \frac{b_j(N^2_k)}{[G_2 : N^2_k]} \cdot \frac{\log \operatorname{tors} H_i(N^1_k)}{[G_1 : N^1_k]} + \frac{b_i(N^1_k)}{[G_1 : N^1_k]} \cdot \frac{\log \operatorname{tors} H_j(N^2_k)}{[G_2 : N^2_k]} \\ & + (\log 2)^{-1} \cdot \frac{\log \operatorname{tors} H_i(N^1_k)}{[G_1 : N^1_k]} \cdot \frac{\log \operatorname{tors} H_j(N^2_k)}{[G_2 : N^2_k]} \Bigg) \\ & + (\log 2)^{-1} \cdot \sum_{i+j=n-1} \frac{\log \operatorname{tors} H_i(N^1_k)}{[G_1 : N^1_k]} \cdot \frac{\log \operatorname{tors} H_j(N^2_k)}{[G_2 : N^2_k]}.\end{aligned}$$

Since all limits superior of the involved terms are finite and non-negative, we can pass to the limit superior by Proposition 1.28, and we get the claimed inequality. $\square$

Remark 3.91. As with Theorem 3.82, the assumptions in the above results could be relaxed further. However, the bounds we have computed in Corollary 3.87 really rely on the fact that the involved homology groups are finitely generated $\mathbb{Z}$-modules.

Corollary 3.92 (degree 1)**.** *Let G_1 and G_2 be residually finite groups of type* FP_2 *together with residual chains $N^1_* \in \mathcal{R}(G_1)$ and $N^2_* \in \mathcal{R}(G_2)$. Then*

$$\widehat{t_1}\left(G_1 \times G_2, N^1_* \times N^2_*\right) \leq \frac{1}{|G_2|} \cdot \widehat{t_1}\left(G_1, N^1_*\right) + \frac{1}{|G_1|} \cdot \widehat{t_1}\left(G_2, N^2_*\right).$$

This becomes an equality if we assume $\widehat{t_1}\left(G_1, N^1_*\right)$ to be a proper limit.

Proof. Dividing the equation in Corollary 3.87 by $[G_1 \times G_2 : N^1_k \times N^2_k]$ gives:

$$\frac{\log \operatorname{tors} H_1(N^1_k \times N^2_k)}{[G_1 \times G_2 : N^1_k \times N^2_k]} = \frac{1}{[G_2 : N^2_k]} \cdot \frac{\log \operatorname{tors} H_1(N^1_k)}{[G_1 : N^1_k]} + \frac{1}{[G_1 : N^1_k]} \cdot \frac{\log \operatorname{tors} H_1(N^2_k)}{[G_2 : N^2_k]}.$$

Passing to the limit superior gives the first claim. If one assumes that $\widehat{t_1}\left(G_1, N^1_*\right)$ is a proper limit, then we get an equality by Proposition 1.28. □

Example 3.93 (arbitrarily small first torsion growth)**.** Consider a residually finite group A together with a residual chain $N_* \in \mathcal{R}(A)$. For $M \in \mathbb{N}_{\geq 1}$, define the group $G := A \times \mathbb{Z}/M$ together with the residual chain $N_* \times 1$. By Corollary 3.92, it satisfies

$$\widehat{t_1}\left(G, N_* \times 1\right) = \frac{1}{M} \cdot \widehat{t_1}\left(A, N_*\right).$$

In particular, if there exists a group A together with a chain N_* with positive first torsion growth, then there exists a group G together with a chain with arbitrarily small positive first torsion growth. Recall that the existence of a finitely presented group A with the above property is unknown (Question 3.26).

Corollary 3.94 (vanishing of torsion growth of direct products)**.** *Let $n \in \mathbb{N}$ and let G_1 and G_2 be residually finite groups of type* FP_{n+1}*. Suppose that*

1. *$G_1, G_2 \in \mathbf{T}_{n-1}$; or*
2. *There exists an $n_0 \in \{0, \ldots, n-1\}$ such that*

$$G_1 \in \mathbf{T}_{n_0} \cap \mathbf{H}_{n_0}(\mathbb{Q}) \quad \text{and} \quad G_2 \in \mathbf{T}_{n-n_0-1} \cap \mathbf{H}_{n-n_0-1}(\mathbb{Q}).$$

If $\widehat{t_m}\left(G_1 \times G_2\right)$ is chain-independent for all $m \in \{1, \ldots, n\}$, then $G_1 \times G_2 \in \mathbf{T}_n$.

Proof. Let $N_*^1 \in \mathcal{R}(G_1)$ and $N_*^2 \in \mathcal{R}(G_2)$ be residual chains. Consider the upper bound inequality in Theorem 3.90.

Ad 1. By assumption, $\widehat{t}_i\,(G_1, N_*^1) = 0$ and $\widehat{t}_j\,(G_2, N_*^2) = 0$ for all $i, j \in \{1, \dots, n-1\}$. Thus, the right-hand side of the inequality vanishes for all degrees up to $n-1$. In degree n, it reduces to

$$\widehat{t}_n\left(G_1 \times G_2, N_*^1 \times N_*^2\right) \leq b_0^{(2)}(G_1) \cdot \widehat{t}_n\left(G_2, N_*^2\right) + b_0^{(2)}(G_2) \cdot \widehat{t}_n\left(G_1, N_*^1\right).$$

The vanishing of the right summand follows by case distinction:

- If $G_1 \times G_2$ is finite, then $\widehat{t}_n\,(G_1 \times G_2, N_*^1 \times N_*^2) = 0$ (Example 3.14).
- If $G_1 \times G_2$ is infinite, where both G_1 and G_2 are infinite, then the zeroth L^2-Betti numbers vanish (Example 3.37), and the right-hand side is zero.
- If $G_1 \times G_2$ is infinite, where G_1 is infinite and G_2 is finite, then we have $b_0^{(2)}(G_1) = 0$ and $\widehat{t}_n\,(G_2, N_*^2) = 0$ and the right-hand side is zero.

Ad 2. Let $m \in \{0, \dots, n\}$. By assumption, for every $i \in \{0, \dots, m\}$, we have that $G_1 \in \mathbf{T}_i \cap \mathbf{H}_i(\mathbb{Q})$ or $G_2 \in \mathbf{T}_{m-i} \cap \mathbf{H}_{m-i}(\mathbb{Q})$. Therefore, every summand in the upper bound vanishes. □

Remark 3.95. Since $\mathbf{T}_*$ is not known to be a bootstrappable property in the sense of Li–Löh–Moraschini–Sauer–Uschold, their bootstrapping theorem does not apply to $\mathbf{T}_*$ as for $\mathbf{H}_*(\mathbb{F})$ (Remark 3.86). Thus the statement of Corollary 3.94 appears to be new.

4. Algorithmic aspects

Every (numerical) group invariant I gives rise to algorithmic questions:

- *Decidability.* Is there a decision process that given a finite presentation $\langle S \mid R\rangle$ of G determines whether $I(G) = 0$ ($\neq 0$, $> c$, $< \infty$, $\ldots$) or not?
- *Computability.* Can we approximate the arising real numbers in a computably controlled way?
- *Realizability.* What real numbers can be realized as $I(G)$ for some group G?

All of the above are classical questions for ordinary numerical group invariants and group properties, which can be regarded as numerical group invariants with values in $\{0, 1\}$. The definition of growth of invariants requires the groups in question to be residually finite, and a priori depends on the choice of a residual chain. Therefore, special care is needed in the precise formulation of the above questions and complicates the study of algorithmic questions about invariant growth. In Chapter 4.1, we review the classical notion of decidability and show that the vanishing problem of the absolute rank gradient is undecidable. In Chapter 4.2, we assemble our earlier results into realizability statements for the rank gradient, torsion growth and Betti number growth. Unfortunately, an interesting result regarding *computability* of torsion growth appears to be out of reach. For an introduction to computability, we instead refer to the survey by Rettinger and Zheng [RZ21]. For an investigation of computability properties of real numbers arising as L^2-Betti numbers of groups, see the article by Löh and Uschold [LU23].

Overview of this chapter.

Convention 4.1 (algorithm)**.** An *algorithm* for us is an algorithm in the sense of a Turing machine (possibly amended by an *oracle*). Turing machines admit a mathematically precise definition which can be found in Rotman's book [Rot09]. It is quite impractical and exceeds the scope of this thesis to work with this description. It is therefore customary to work with an intuitive definition of an algorithm as a well-defined computational procedure with finitely many rules. *Church's thesis* roughly states that most "reasonable" approaches towards the notion of an algorithm lead to equivalent definitions.

4.1. Decidability

It is a standard fact in group theory that every group admits a presentation, which is a concise way to encode its structure via generators and relations. A group is called *finitely presented* if it has a finite set of generators and relations. These groups are considered “small” in the sense that their entire structure is determined by finite data. It is natural to ask whether there exists an algorithm that, given a finite presentation, decides whether the corresponding group satisfies a certain property. Such questions are known as *decision problems*. However, as a general rule, most interesting group-theoretic properties are *undecidable*. That is to say, there exists no algorithm that solves the decision problem for all finite presentations. A formal introduction into decision problems and their connection to Turing machines can be found in Rotman’s book [Rot09, Chapter 12].

Metaconjecture 4.2. *The following algorithmic problem is undecidable:*

> *Let $n \geq 1$, and let $\mathbb{F}$ be a field. Given a finite presentation $\langle S \mid R\rangle$, decide whether $\langle S \mid R\rangle$ has vanishing torsion growth (resp. $\mathbb{F}$-Betti number growth) in degree n or not.*

This question in the above form is not well-phrased. We have defined torsion growth and Betti number growth of residually finite groups with respect to a chosen residual chain. However, we have seen that for the rank gradient, there is a way to get rid of the dependence on the residual chain. I believe this should be the general strategy to approach and formulate the above Metaconjecture. One straightforward way of showing that a group property is undecidable, is achieved by the *Adian–Rabin theorem.*

Definition 4.3 (Markov property)**.** A subclass P of the class of finitely presentable groups is a *Markov property* if the following properties hold:

- The class P is closed under taking isomorphisms.
- *Positive witness.* The class P is non-empty.
- *Negative witness.* There exists a finitely presentable group that is not isomorphic to a subgroup of an element of P.

Theorem 4.4 (Adian–Rabin theorem [Rot12, Theorem 12.32])**.** *Let P be a Markov property of finitely presentable groups. Then the following algorithmic problem is undecidable:*

> *Given a finite presentation $\langle S \mid R\rangle$, decide whether $\langle S \mid R\rangle$ lies in P or not.*

Corollary 4.5 (residual finiteness is undecidable)**.** *The following algorithmic problem is undecidable:*

> *Given a finite presentation $\langle S \mid R\rangle$, decide whether $\langle S \mid R\rangle$ is residually finite or not.*

Proof. By the Adian–Rabin theorem, it suffices to show that being residually finite is a Markov property. Closure under isomorphisms is clear. A positive witness is given by $\mathbb{Z}$. A negative witness is given by any finitely presentable non-residually finite group, e.g., $\mathrm{BS}(2,3)$ (Example 1.5). Such a group does not embed into a residually finite group, by Proposition 1.7. □

We will now prove the undecidability of the vanishing problem of the absolute rank gradient. For this, we use a standard technique based on the *Novikov–Boone–Britton theorem.*

Theorem 4.6 (Novikov–Boone–Britton theorem [Rot09, Theorem 12.8])**.** *There exists a finitely presented group Λ with unsolvable word problem.*

Theorem 4.7 (undecidability of vanishing rank gradient)**.** *The following algorithmic problem is undecidable:*

> *Given a finite presentation $\langle S \mid R\rangle$, decide whether $\langle S \mid R\rangle$ has vanishing rank gradient or not.*

Proof. We perform essentially the same construction that Fournier-Facio, Löh and Moraschini have used to exhibit undecidability results for bounded cohomology [FLM24, Construction 8.8]. By the Novikov–Boone–Britton theorem 4.6 there exists a finitely presented group $\Lambda = \langle S \mid R\rangle$ with unsolvable word problem. By the proof of the Adian–Rabin theorem [Rot09, Lemma 12.31], there exists an algorithm

$$\begin{aligned} \text{words over } S &\to \text{finite presentations} \\ w &\mapsto \Lambda_w := \langle S_w \mid R_w\rangle \end{aligned}$$

such that w represents the neutral element of Λ if and only if $\langle S_w \mid R_w\rangle \cong 1$. This construction can be refined: we may assume that there is an algorithm

$$\begin{aligned} \text{words over } S &\to \text{words over } S_w \\ w &\mapsto \overline{w} \end{aligned}$$

such that $\overline{w}$ has infinite order in Λ_w if w does not represent the neutral element in Λ. For every word w over S, define $\Gamma_w := \Lambda_w * \mathbb{Z}$ (notice that we can determine the presentation of Γ_w from the presentation of Λ_w in a computably controlled way). By construction, we have:

- If w represents the neutral element of Λ, then

$$\Gamma_w \cong \Lambda_w * \mathbb{Z} \cong 1 * \mathbb{Z} \cong \mathbb{Z}.$$

In particular, $\mathrm{RG}(\Gamma_w) = 0$ by Example 1.60.

- If w does *not* represent the neutral element of Λ, then Λ_w is non-trivial and contains an element $\overline{w}$ of infinite order, so Γ_w is a non-elementary free product, i.e., a free product $A * B$, where A and B are non-trivial and we do not have $A \cong \mathbb{Z}/2 \cong B$. In particular, by Corollary 1.65 we have that $\mathrm{RG}(G) > 0$.

Therefore, if one could decide the vanishing problem of the absolute rank gradient, this would imply solvability of the word problem in Λ, which contradicts the choice of Λ. $\square$

Remark 4.8 (undecidability for homology growth)**.**

4.2. Realizability

The study of the range of values is specific to *numerical* (group) invariants. The fundamental question is: given $r \in \mathbb{R}$, does there exist a group G (in some class of groups) such that $I(G) = r$? In Chapter 3.2 we saw that the task of computing homology growth of right-angled Artin groups amounts to computing the homology groups of the underlying flag complex. We have seen that flag complexes can be constructed to have a prescribed sequence of Abelian groups as their homology groups. We can get finer realizability by taking direct products with finite groups.

Example 4.9 (absolute rank gradient)**.** As previously observed by Pappas, we have that for every $r \in \mathbb{Q}_{\geq 0} \cup \{-M^{-1} \mid M \in \mathbb{N}_{\geq 2}\}$ there exists a finitely presented group G such that $\mathrm{RG}(G) = r$ [Pap13, Proposition 2.2]:

- If $r = 0$, put $G := \mathbb{Z}$ such that $\mathrm{RG}(G) = r$ by Example 1.60.

- If $r = -M^{-1}$ for $M \in \mathbb{N}_{\geq 2}$, put $G := \mathbb{Z}/M$ such that $\mathrm{RG}(G) = r$ by Example 1.64.

- If $r > 0$, then we may write $r = \frac{N}{M}$ with $N \in \mathbb{N}_{\geq 1}$ and $M \in \mathbb{N}_{\geq 2}$. Put $G := F_{N+1} \times \mathbb{Z}/M$ and consider the index M normal subgroup $H := \ker(G \twoheadrightarrow \mathbb{Z}/M) \cong F_{N+1}$ of G. By Lemma 1.50, we get

$$\mathrm{RG}(G) = \frac{\mathrm{RG}(H)}{[G : H]} = \frac{N}{M}.$$

The following problem is still open:

> *Does every non-negative real number $r \in \mathbb{R}_{\geq 0}$ arise as the absolute rank gradient of a finitely generated group?*

Recall that in Chapter 3.2.3 we have seen that flag complexes realize any given sequence of Abelian groups as their homology groups. Combining this with the computations of homology growth of right-angled Artin groups in Chapter 3.2.2, we can realize (almost) arbitrary sequences of (logarithms of) natural numbers as the torsion growths and Betti number growths of a finitely presented group.

Corollary 4.10 (torsion growth and $\mathbb{Q}$-Betti number growth)**.** *Let $(b_i)_{i\geq 1}$ and $(t_i)_{i\geq 2}$ be sequences with values in $\mathbb{N}$ and $\mathbb{N}_{\geq 1}$, respectively, such that only finitely many values are non-zero. Then there exists a finitely presented residually finite group G such that for all $n \in \mathbb{N}$*

$$\widehat{b_n}\,(G;\mathbb{Q}) = \begin{cases} 0 & \textit{if } n = 0, \\ b_n & \textit{if } n \geq 1, \end{cases}$$

and

$$\widehat{t_n}\,(G) = \begin{cases} 0 & \textit{if } n \in \{0,1\}, \\ \log t_n & \textit{if } n \geq 2. \end{cases}$$

Here, $\widehat{b_n}\,(G;\mathbb{Q})$ and $\widehat{t_n}\,(G)$ are given by chain-independent proper limits for all $n \in \mathbb{N}$.

Proof. Consider the sequence of finitely generated Abelian groups $A_i := \mathbb{Z}^{b_{i+1}} \oplus \mathbb{Z}/t_{i+1}$ for all $i \geq 2$. Then by Corollary 3.68, there exists a finite connected flag complex L such that $H_i(L;\mathbb{Z}) \cong_{\mathbb{Z}} A_i$ for all $i \geq 2$. We may assume that L has (b_1+1)-many connected components by taking a disjoint union of L with b_1-many disjoint points. For $G := A_L$ we thus have for all $i \geq 2$

- $\widehat{b_1}\,(G;\mathbb{Q}) = \widetilde{b}_0(L;\mathbb{Q}) = b_1$;
- $\widehat{b_i}\,(G;\mathbb{Q}) = b_{i-1}(L;\mathbb{Q}) = \dim_{\mathbb{Q}} A_{i-1} = b_i$;
- $\widehat{t_i}\,(G) = \log \operatorname{tors} H_{i-1}(L;\mathbb{Z}) = \log \operatorname{tors} A_{i-1} = \log t_i$. ☐

Similarly, one can show that there exists a finitely presented group G such that $G \in \mathbf{T}_\infty$ and for all $n \in \mathbb{N}$

$$\widehat{b_n}\,(G;\mathbb{F}_p) = \begin{cases} 0 & \text{if } n = 0, \\ b_i & \text{if } n \geq 1. \end{cases}$$

Corollary 4.11 (torsion growth and $\mathbb{Q}$-Betti number growth, refined)**.** *Let $(b_i)_{i\geq 1}$ be sequence with values in $\mathbb{Q}_{\geq 0}$ and $(t_i)_{i\geq 2}$ with values in $\frac{\log \mathbb{N}_{\geq 1}}{\mathbb{N}_{\geq 1}}$, such that only finitely many values are non-zero. Then there exists a finitely presented residually finite group G together with a residual chain $N_* \in \mathcal{R}(G)$ such that for all $n \in \mathbb{N}$*

$$\widehat{b_n}\,(G, N_*; \mathbb{Q}) = \begin{cases} 0 & \text{if } n = 0, \\ b_n & \text{if } n \geq 1, \end{cases}$$

and

$$\widehat{t_n}\,(G, N_*) = \begin{cases} 0 & \text{if } n \in \{0,1\}, \\ t_n & \text{if } n \geq 2. \end{cases}$$

Here, $\widehat{b_n}\,(G, N_*; \mathbb{Q})$ and $\widehat{t_n}\,(G, N_*)$ are given by proper limits for all $n \in \mathbb{N}$.

Proof. By passing to the greatest common divisor, we may assume without loss of generality that there exists an $M \in \mathbb{N}_{\geq 2}$ such that for all $i \geq 1$

$$b_i = \frac{\beta_i}{M} \text{ with } \beta_i \in \mathbb{N},$$

and for all $i \geq 2$

$$t_i = \frac{\log \tau_i}{M} \text{ with } \tau_i \in \mathbb{N}_{\geq 1}.$$

Applying Corollary 4.10 to the sequences $(\beta_i)_{i\geq 1}$ and $(\tau_i)_{i\geq 2}$, we get a finitely presented residually finite group A such that

$$\widehat{b_n}\,(A; \mathbb{Q}) = \begin{cases} 0 & \text{if } n = 0, \\ \beta_n & \text{if } n \geq 1, \end{cases}$$

and

$$\widehat{t_n}\,(A) = \begin{cases} 0 & \text{if } n \in \{0,1\}, \\ \log \tau_n & \text{if } n \geq 2. \end{cases}$$

For the group A, torsion growth and Betti number growth are given by chain-independent proper limits. Consider an arbitrary residual chain $N_*^A \in \mathcal{R}(A)$. By Example 1.49 we have that for the residual chain $N_* := N_*^A \times 1$ in G we have

$$\widehat{b_n}\,(G, N_*; \mathbb{Q}) = \begin{cases} 0 & \text{if } n = 0, \\ b_n & \text{if } n \geq 1, \end{cases}$$

and

$$\widehat{t_n}\,(G, N_*) = \begin{cases} 0 & \text{if } n \in \{0,1\}, \\ t_n & \text{if } n \geq 2, \end{cases}$$

both given by proper limits. □

Example 4.12 (L^2-Betti numbers)**.** Let $n \in \mathbb{N}_{\geq 2}$. By additivity of $b_n^{(2)}$ under free products and the fact that $b_n^{(2)}(G) = |G|^{-1}$ for a finite group G, it is easily deduced that every non-negative rational number arises as the L^2-Betti number of a finitely presented group. Grabowski, and independently, Pichot, Schick and Zuk extended this to all non-negative real numbers [Gra14, Theorem 1.3][PSZ15]. However, the following variant of *Atiyah's problem* is still open:

Is the L^2-Betti number of every finitely generated torsion-free group an integer?

A. Introduction to Bass–Serre theory

One of the prototypical results of Geometric group theory states that free groups can be characterized as the groups that admit a free action on a tree. The Nielsen–Schreier theorem is an immediate corollary and is notoriously hard to prove purely algebraically. The above result also comes with a procedure that allows to construct a concrete free generating set of a group acting freely on a tree.

Bass–Serre theory generalizes the above question and studies groups G that admit a (not necessarily free) action on a tree T. It turns out that the quotient graph $G \setminus T$ of such an action encodes a way to decompose G into smaller pieces given by the vertex and edge stabilizers. This decomposition has as basic building blocks amalgamated products and HNN-extensions. In this chapter, we will survey the main theorem of Bass–Serre theory, that will allow us to decompose finite index normal subgroups of free products, which was the key step in proving Theorem 1.52. Moreover, one can also give an alternative proof of the quantitative Nielsen–Schreier theorem, which we used several times throughout the thesis.

We closely follow the exposition given in Serre's book [Ser80].

Definition A.1 (Serre's notion of a graph [Ser80, §2.1, Definition 1])**.** A *graph* $Y = (V, E)$ consists of a set V of *vertices* and a set E of *edges* and two maps

$$E \to V \times V,\ e \mapsto \big(o(e), t(e)\big) \text{ and } E \to E,\ e \mapsto \widetilde{e},$$

such that for every $e \in E$ we have that

$$\widetilde{\widetilde{e}} = e\ ,\ \widetilde{e} \neq e \text{ and } o(e) = t(\widetilde{e}).$$

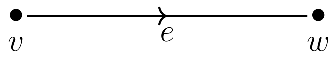
This coincides with the common notion of a simplicial graph (with a chosen orientation) via the following interpretation: for every edge $e \in E$, $o(e)$ is the *origin of* e, $t(e)$ the *terminus of* e, and $\widetilde{e}$ is the same edge with opposite orientation.

Convention A.2. In this chapter, whenever we say that Y is a *graph*, we shall mean that $Y = (V, E, o, t, \widetilde{\cdot})$ in the sense of the above Definition A.1. We suppress the underlying data from the notation and write $V(Y)$ and $E(Y)$ to denote the vertex set and edge set of Y, respectively.

Definition A.3 (graph of groups [Ser80, §4.4, Definition 8])**.** A *graph of groups* $(\mathcal{G}, Y)$ consists of a graph Y, families of groups $\{G_v \mid v \in V(Y)\}$ and $\{G_e \mid e \in E(Y)\}$ such that $G_e = G_{\widetilde{e}}$, as well as injective homomorphisms $G_e \to G_{t(e)}, a \mapsto a^e$ for every $e \in E(Y)$.

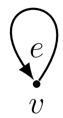
The two basic examples are given by *segment* and *loop* graphs of groups.

Example A.4 (segment graph of groups)**.** Consider the oriented graph Y given by a *segment*



The data of a graph of groups on Y is given by two vertex groups G_v and G_w and an edge group G_e, together with two injective homomorphisms $G_e \hookrightarrow G_v$ and $G_e \hookrightarrow G_w$.

Example A.5 (loop graph of groups)**.** Consider the *loop graph*



The data of a graph of groups on Y is given by a vertex group G_v and an edge group G_e together with two injective homomorphisms $G_e \hookrightarrow G_v$.

Definition A.6 (fundamental group [Ser80, §5.1])**.** Let $(\mathcal{G}, Y)$ be a graph of groups, let T be a maximal subtree of Y. The *fundamental group* $\pi_1(\mathcal{G}, Y, T)$ *of* $(\mathcal{G}, Y)$ *at* T is given by the following presentation:

- generators:
  - the generators of G_v for $v \in V(Y)$;
  - g_y for $y \in E(Y)$.
- relations:
  - the relations of G_v for $v \in V(Y)$;
  - $g_y a^y g_y^{-1} = a^{\widetilde{y}}$ and $g_{\widetilde{y}} = g_y^{-1}$ for $y \in E(Y)$ and $a \in G_y$;
  - $g_y = 1$ for $y \in E(T)$.

Importantly, it turns out that this definition is *independent* of the chosen maximal subtree T of Y [Ser80, §5.1, Proposition 20].

Example A.7 (fundamental group of segment [Ser80, p.43, Example 2)])**.** Let $(\mathcal{G}, Y)$ be a graph of groups on a segment like in the example above. Then the unique maximal tree is $T = Y$. We get for the fundamental group a presentation

$$\langle G_v, G_w \mid a^e = a^{\widetilde{e}}, a \in G_e \rangle \cong G_v *_{G_e} G_w,$$

which corresponds to the amalgamated product of G_v and G_w over the common subgroup G_e.

Example A.8 (fundamental group of loop [Ser80, p.43, Example 3)]). Let $(\mathcal{G}, Y)$ be a graph of groups on a loop like in the example above. Then the unique maximal tree is given by the single vertex $T = v$. The fundamental group has presentation (renaming $t := g_e$)

$$\langle G_v, t \mid ta^e t^{-1} = a^{\widetilde{e}}, a \in G_e \rangle \cong G_v *_{G_e},$$

which corresponds to the HNN-extension of G_v by the automorphism of subgroups isomorphic to G_e induced by the two injections of G_e into G_v.

Theorem A.9 (Bass–Serre tree [Ser80, §5.3, Theorem 12]). *Let $(\mathcal{G}, Y)$ be a graph of groups with Y connected and non-empty, and let T be a maximal tree in Y. Then there exists a* Bass–Serre tree $\widetilde{X}$ *given by the following data:*

- *a graph $\widetilde{X} := \widetilde{X}(\mathcal{G}, Y, T)$;*
- *an action of $\pi := \pi_1(\mathcal{G}, Y, T) \curvearrowright \widetilde{X}$;*
- *a graph map $p : \widetilde{X} \to Y$ that induces an isomorphism $\pi \setminus \widetilde{X} \cong_{\text{graph}} Y$;*
- *sections $V(Y) \to V(\widetilde{X})$ and $E(Y) \to E(\widetilde{X})$ (denoted $v \mapsto \widetilde{v}$ and $e \to \widetilde{e}$, respectively);*

such that following conditions are satisfied:

- *Given $v \in V(Y)$, the stabilizer $\pi_{\widetilde{v}}$ of $\widetilde{v}$ is isomorphic to G_v;*
- *Given $e \in E(Y)$, the stabilizer $\pi_{\widetilde{e}}$ of $\widetilde{e}$ is isomorphic to the image of G_e in $G_{t(e)}$;*
- *$\widetilde{X}$ is a tree.*

Such a Bass–Serre tree is unique with that property.

Theorem A.10 (main theorem of Bass–Serre theory [Ser80, §5.4, Theorem 13]). *Suppose G acts on a connected non-empty graph X. Let $Y = G \setminus X$ and let T be a maximal tree in Y. Define $(\mathcal{G}, Y, T)$ and $\widetilde{X}$ as above. Then the following are equivalent:*

1. *X is a tree.*
2. *X is isomorphic to $\widetilde{X}$.*
3. *G is isomorphic to $\pi_1(\mathcal{G}, Y, T)$.*

Corollary A.11 (groups acting on trees)**.** *Suppose G acts on a connected non-empty tree X. Let $Y = G \setminus X$ and let T be a maximal tree in Y. Then, G has the following presentation:*

- *generators: G_v for $v \in V(Y)$, g_e for $e \in E(Y)$;*
- *relations: $g_e a^e g_e^{-1} = a^{\widetilde{e}}$, $g_{\widetilde{e}} = g_e^{-1}$ for $e \in E(Y)$ and $g_e = 1$ if $e \in E(T)$.*

Proof. This follows immediately from the existence of a Bass–Serre tree (Theorem A.9), the main theorem of Bass–Serre theory (Theorem A.10), and the definition of the fundamental group of a graph of groups (Definition A.6). □

Theorem A.12 (quantitative Nielsen–Schreier theorem)**.** *Let $m \in \mathbb{N}_{\geq 1}$, and let $G = F_m$ be "the" free group of rank m. Let $H \subseteq F_m$ be a finite index subgroup. Then H is free of rank*

$$(m-1) \cdot [G : H] + 1.$$

Proof. Notice that G is the fundamental group of the graph of groups $(\mathcal{G}, Y)$ consisting of one vertex and m edges with trivial vertex and edge groups. Hence G acts freely on its Bass–Serre tree $\widetilde{X}$ with quotient graph Y. The subgroup H also acts freely on $\widetilde{X}$. By the main theorem of Bass–Serre theory A.10, H is isomorphic to the fundamental group of the graph of groups associated to $H \curvearrowright \widetilde{X}$. For this, it suffices to understand the structure of the quotient graph $Y = H \setminus \widetilde{X}$. Note that it has trivial vertex and edge groups, since the action is free. By the orbit-stabilizer theorem

- $|V(Y)| = \left|H \setminus V(\widetilde{X})\right| = [G : H] \cdot |G \setminus V(T)| = [G : H]$, and
- $|E(Y)| = |H \setminus E(T)| = [G : H] \cdot |G \setminus E(T)| = [G : H] \cdot m.$

Every maximal subtree T of Y has

$$|E(T)| = |V(Y)| - 1 = [G : H] - 1,$$

and therefore

$$|E(Y) \setminus E(T)| = |E(Y)| - |E(T)| = [G : H] \cdot m - ([G : H] - 1) = (m-1) \cdot [G : H] + 1.$$

These edges $e \in Y \setminus T$ are in bijection with the free generators of H. Thus, H is a free group of rank $(m-1) \cdot [G : H] + 1$. □

Theorem A.13 (subgroups of free products)**.** *Let G_1 and G_2 be groups, let $G := G_1 * G_2$ be the free product. Let $N \trianglelefteq G$ be a finite index normal subgroup of index* *d. For $l \in \{1,2\}$, we have that $N_l := N \cap G_l$ are normal subgroups in G_l, say of index d_l. Then, N has the following decomposition:*

$$N \cong \underbrace{N_1 * \ldots * N_1}_{\frac{d}{d_1}-many} * \underbrace{N_2 * \ldots * N_2}_{\frac{d}{d_2}-many} * F,$$

where F is a free group of rank $d - \frac{d}{d_1} - \frac{d}{d_2} + 1$.

Proof. We make use of the machinery of Bass–Serre theory. We know that G acts on its Bass–Serre tree X, with trivial edge stabilizers and vertex stabilizers G_1 or G_2. We get an induced action of N on X. By the main theorem of Bass–Serre theory, we get that N is isomorphic to the fundamental group of the graph of groups associated to the action $N \curvearrowright X$. It suffices to understand the structure of the quotient graph $Y = N \setminus X$ and its vertex and edge groups. Since all edge stabilizers of the group action are trivial, all the edge groups in Y will be trivial. In particular, the fundamental group is a free product of the vertex groups, together with new generators $E(Y) \setminus E(T)$, subject to no relations. Notice that this corresponds to taking the free product with a free group of rank $|E(Y) \setminus E(T)|$, that we will determine later.
Since vertices in Y correspond to n-orbits in X, we count the n-orbits of G_1-vertices. The same follows for G_2. By normality of N, we see the following: let $N \cdot gG_1$ be an n-orbit. Then we have

$$N \cdot (gG_1) = (Ng)G_1 = (gN)G_1 = g(NG_1) = g(G_1N),$$

hence there is a bijection of sets between $N \cdot gG_1$ and $G/(G_1N)$. This allows us to count the orbits:

$$|G/(G_1N)| = [G : G_1N] = \frac{[G:N]}{[G_1N:N]} = \frac{[G:N]}{[G:G_1 \cap N]} = \frac{d}{d_1}.$$

What are the n-stabilizers of G_1 vertices in X? We know that the G-stabilizers are given by (conjugates of) G_1, so in particular we obtain for every G_1-vertex $gG_1 \in G/G_1$:

$$\mathrm{stab}_N(gG_1) = \mathrm{stab}_G(gG_1) \cap N = gG_1g^{-1} \cap N = g(G_1 \cap N)g^{-1},$$

that is isomorphic to N_1. The same follows for the G_2-vertices.

We now count the edges in Y, which amounts to counting the n-orbits of edges in X. Recall that the set of edges is G, where an edge $g \in G$ connects vertices gG_1 and gG_2. Let $N \cdot g$ be an n-orbit of the edge $g \in G$. Since we have $N \cdot g = Ng = gN$, the set of n-orbits is in bijection with G/N, that has cardinality d. Notice that Y does not have loops, since edges in X connect G_1- and G_2-vertices, that have different orbits and therefore do not get identified in Y. For a similar reason, there are no double edges. This allows us to determine the number of edges of a maximal tree to be $\frac{d}{d_1} + \frac{d}{d_2} - 1$. Therefore, the number of edges that lie outside of a given maximal tree is $d - \frac{d}{d_1} - \frac{d}{d_2} + 1$. Every such edge contributes a generator subject to no relations, thus this is precisely the rank of the free group F. $\square$

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Table of notation

$\mathrm{cd}(G)$	cohomological dimension of G
$\operatorname{char}\mathbb{F}$	characteristic of the field $\mathbb{F}$
$\chi(G)$	Euler characteristic
$\delta(G)$	deficiency of G
$\mathbb{F}$	field
$\mathbb{F}_p$	finite field of characteristic p
F_n, F_∞, F	topological finiteness properties
FP_n, FP_∞, FP	homological finiteness properties
$\Gamma * \Lambda$	join of the graphs Γ and Λ
$\Gamma \sqcup \Lambda$	disjoint union of the graphs Γ and Λ
$\mathrm{gd}(G)$	geometric dimension of G
$b_n^{(2)}(G)$	n-th L^2-Betti number of G
$\log$	natural logarithm
$\widehat{t_n}\,(G, N_*)$	n-th torsion growth of G wrt. N_*
$\mathrm{Mat}(m \times n; R)$	set of $m \times n$ matrices with R coefficients
$\mathbb{T}^n$	n-torus
$\mathbf{T}_n$, $\mathbf{T}_\infty$	torsion growth vanishing classes
$\mathcal{F}(G)$	set of finite index subgroups of G
$\mathrm{BS}(m, n)$	Baumslag–Solitar group
$\mathrm{flag}(\Gamma)$	flag completion of Γ
$\mathrm{K}(G, 1)$	classifying space for G
Sym_r	symmetric group over r elements

$\operatorname{mult}_p A$ .... multiplicativity of p in A

$\mathbb{N}$ .... natural numbers

$\overline{\mathbb{R}}$ .... extended reals

$\pi_1(X, x_0)$ .... fundamental group of X wrt. basepoint x_0

$\langle S \mid R \rangle$ .... group generated by S with relations R

$\mathbb{R}P^2$ .... real projective space

$\mathbb{Q}$ .... rational numbers

$\mathbb{R}$ .... real numbers

$\mathcal{R}(G)$ .... set of residual chains of G

Res^G_H .... restriction functor from G to H

$\operatorname{RG}(G, N_*)$ .... rank gradient of G wrt. N_*

$\operatorname{rk}_{\mathbb{Z}} A$ .... rank of A

$\operatorname{rk}_{\mathbb{Z}}$ .... rank of a $\mathbb{Z}$-module

$\operatorname{Sal}_\Gamma$ .... Salvetti complex of Γ

Σ_g .... orientable surface of genus g

$\mathbb{S}^n$ .... n-sphere

$\operatorname{tors} A$ .... size of torsion submodule of A

$\operatorname{Tor}_1^R(A, B)$ .... Tor functor of A and B

$\widetilde{b}_n(X; \mathbb{F})$ .... reduced $\mathbb{F}$-Betti number of X

$\widetilde{e}$ .... oppositely oriented edge of e

$\widetilde{H}_n(X; \mathbb{Z})$ .... reduced homology group of X

$\mathbb{Z}G$ .... group ring of G

$\mathbb{Z}$ .... integers

A_Γ .... right-angled Artin group associated to Γ

$b_n(G;\mathbb{F}_p)$	n-th mod p (or $\mathbb{F}_p$-) Betti number of G
$b_n(G;\mathbb{Q})$	n-th Betti number of G
BG	classifying space for G
$d(G)$	rank of G
F_m	free group of rank m
$G \curvearrowright X$	left G-action on X
$H_n(G;A)$	group homology of G with coefficients A
$M(A,n)$	Moore space
$o(e)$	origin of e
S_g	non-orientable surface of genus g
$T(A)$	torsion submodule of A
$t(e)$	terminus of e

Selbständigkeitserklärung

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in § 26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, den 11.08.25

Ort, Datum

Rusakov

Unterschrift Illja Rusakov